THE TENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Tenth Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Monday and Tuesday, December 28–29, 1903. The following forty-eight members attended the four sessions:

Mr. Joseph Allen, Dr. Grace Andrews, Professor R. C. Archibald, Dr. C. L. Bouton, Professor Joseph Bowden, Professor E. W. Brown, Professor F. N. Cole, Miss E. B. Cowley, Dr. W. S. Dennett, Dr. Otto Dunkel, Professor F. C. Ferry, Dr. William Findlay, Professor B. F. Finkel, Professor T. S. Fiske, Dr. A. S. Gale, Dr. W. A. Granville, Dr. G. H. Hallett, Dr. J. G. Hardy, Professor J. Harkness, Dr. C. N. Haskins, Professor L. I. Hewes, Mr. E. A. Hook, Professor L. S. Hulburt, Dr. E. V. Huntington, Professor J. I. Hutchinson, Mr. L. L. Jackson, Dr. Edward Kasner, Professor C. J. Keyser, Mr. L. L. Locke, Dr. Emory McClintock, Professor James Maclay, Dr. Isabel Maddison, Mr. C. L. E. Moore, Professor F. Morley, Dr. L. I. Neikirk, Professor W. F. Osgood, Dr. I. E. Rabinovitch, Professor L. W. Reid, Dr. Arthur Schultze, Professor Charlotte A. Scott, Professor P. F. Smith, Dr. C. E. Stromquist, Professor H. W. Tyler, Mr. C. B. Upton, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Dr. E. B. Wilson, Mr. J. E. Wright.

The President of the Society, Professor Thomas S. Fiske, occupied the chair, being relieved by Vice-President Professor W. F. Osgood. The Council announced the election of the following persons to membership in the Society: Mr. R. F. Deimel, Columbia University; Mr. C. S. Forbes, Columbia University; Professor O. T. Geckeler, Georgia School of Technology; Mr. E. A. Hook, Columbia University; Mr. L. A. Martin, Jr., Stevens Institute of Technology; Miss Virginia Ragsdale, New York City; Mr. S. E. Rasor, Ohio State University; Mr. A. E. Young, Purdue University; Mr. J. E. Wright, Bryn Mawr College. Six applications for membership in the Society were received.

Reports were presented by the Treasurer, the Auditing Committee, and the Librarian. These reports have already ap-

peared in the Annual Register, published in January. During the past year the membership of the Society has shown an unusual increase, the net gain being 54, making the total membership 455 on January 1, 1904. The number of papers presented at the meetings of the Society in 1903 was 127; the total attendance at all the meetings was 222; 136 members attended at least one meeting during the year. The library now contains over 1,300 bound volumes, an increase of some 300 since last year. The exchange list of the BULLETIN and the *Transactions* includes 154 journals. A catalogue of all journals in the library and of accessions other than journals received during the year is printed in the Annual Register. The Treasurer's report shows a balance of $2,573.07 on hand, December 23, 1903. Of this balance $1,302.50 is credited to the life-membership fund. The number of life members of the Society is now 30.

At the annual election, which closed on Tuesday morning, the following officers and other members of the Council were chosen:

Vice-Presidents,	Professor OSKAR BOLZA,
	Professor J. M. VAN VLECK.
Secretary,	Professor F. N. COLE.
Treasurer,	DR. W. S. DENNETT.
Librarian,	Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor ALEXANDER ZIWET,
Professor D. E. SMITH.

Members of the Council to serve until December, 1906,

Professor MAXIME BÔCHER,	Professor M. B. PORTER,
Professor FLORIAN CAJORI,	Professor J. H. TANNER.

A committee consisting of the President and Professor Osgood was appointed to arrange for the publication in book form of the courses of lectures delivered by Professor E. B. Van Vleck, Professor H. S. White and Professor F. S. Woods before the Boston Colloquium, in September, 1903. A committee was also appointed to consider the question of the financial support of the *Transactions* after the expiration of the present university subscriptions.

A pleasant social feature of the occasion was a dinner on Monday evening, which was attended by about thirty members of the Society.

The following papers were read at the annual meeting :

(1) Dr. E. V. HUNTINGTON : "A set of independent postulates for the algebra of logic (second paper)."

(2) Dr. J. G. HUN : "On certain invariants of two triangles."

(3) Dr. O. D. KELLOGG : "Note on Cauchy's integral."

(4) Professor J. I. HUTCHINSON : "On certain automorphic functions."

(5) Professor W. F. OSGOOD : "On a gap in the usual presentation of Weierstrass's theory of functions."

(6) Dr. E. V. HUNTINGTON : "Third complete set of postulates for the theory of positive integers."

(7) Dr. E. V. HUNTINGTON : "Second complete set of postulates for the theory of magnitudes or positive real quantities."

(8) Dr. W. B. FITE : "On some properties of groups whose orders are powers of a prime."

(9) Professor E. J. WILCZYNSKI : "On ruled surfaces whose flecnode curve intersects every generator in two coincident points."

(10) Professor VIRGIL SNYDER : "Complete enumeration of sextic scrolls having a rectilinear directrix."

(11) Professor F. MORLEY : "On the triplicity of 3-points in a plane."

(12) Mr. C. L. E. MOORE : "Classification of surfaces of singularities in the quadratic spherical complex."

(13) Mr. L. D. AMES : "On the theorem of analysis situs relating to the division of the plane or of space by a closed curve or surface."

(14) Mr. W. B. FORD : "On the function defined by a Maclaurin series."

(15) Professor P. F. SMITH : "Linear transformation of a quadratic form into itself."

(16) Dr. E. B. WILSON : "Projective and metric geometry."

(17) Mr. C. H. SISAM : "On the depiction of the lines of a special linear complex on the points of space."

(18) Dr. EDWARD KASNER : "Investigations on isothermal systems."

Dr. Hun's paper was communicated to the Society through Dr. Kasner. Mr. Sisam was introduced by Professor Snyder.

In the absence of the authors, Mr. Ames's paper was read by Professor Osgood, Dr. Fite's paper by Professor Hutchinson, and the papers of Dr. Hun, Dr. Kellogg, Professor Wilczynski, Professor Snyder, Mr. Ford, and Professor Smith were read by title. The papers of Dr. Kellogg, Professor Osgood, and Mr. Ames will appear in the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In the postulates of this paper Dr. Huntington expresses a necessary and sufficient condition that a class K, with a relation $\prec$, shall be equivalent to the class of logic classes with the relation of inclusion.*

1. $a \prec a$ whenever a is in the class.
2. If $a \prec b$ and $b \prec a$, then $a = b$.
3. If $a \prec b$ and $b \prec c$, then $a \prec c$.
4. There is an element z such that $z \prec a$ for every a.
5. There is an element i such that $a \prec i$ for every a.
6. If $a \neq b$, there is an element s such that 1) $a \prec s$; 2) $b \prec s$; 3) if $a \prec y$ and $b \prec y$ then $s \prec y$.
7. If $a \neq b$, there is an element p such that 1) $p \prec a$; 2) $p \prec b$; 3) if $x \prec a$ and $x \prec b$ then $x \prec p$.
8. If the elements z and i in 4 and 5 exist and are unique, then for every element a there is an element a' such that 1) if $x \prec a$ and $x \prec a'$ then $x \prec z$; 2) if $a \prec y$ and $a' \prec y$, then $i \prec y$.

Definition $a \oplus b$. If $a \neq b$, $a \oplus b = s$ (see postulate 6); if $a = b$, $a \oplus a = a$.

Definition $a \odot b$. If $a \neq b$, $a \odot b = p$ (see postulate 7); if $a = b$, $a \odot a = a$.

9. If the element z in 4 exists uniquely, and if the elements s and p in 6 and 7 exist and are uniquely determined by a and b; and if, further, $b \odot c = z$; then $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

These nine postulates are shown to be independent.

2. With two plane curves, one of the nth order $\alpha_x^n = 0$ and the other of the nth class $\alpha_\xi^n = 0$, there is associated the connex $\alpha_y \alpha_\eta \alpha_\omega^{n-1} = 0$. Some of the invariants of the two curves may be expressed by means of the invariants of this connex. In Dr. Hun's paper the curves considered are a 3-line and a 3-point. A few of the simpler invariant relations of the two triangles are expressed in terms of the invariants of the connex.

*Cf. Schroeder, Algebra der Logik, and papers by C. S. Peirce.

From the forms of these relations the truth of certain well-known theorems may be seen immediately, *e. g.*, that two conics apolar to a conic are circumscribed to a conic. By this method it is also shown that the relation between the fixed triangle of a normal collineation and a Pasch triangle belonging to that collineation is a mutual one. Toward the end of the article some attention is devoted to a conjugate 4-point of a 3-line, and to the collineations leaving fixed three points of the 4-point. The paper will appear in the January number of the *Transactions*.

4. The paper by Professor Hutchinson discusses the functions belonging to the group whose fundamental region is a triangle having angles $\pi/3$, $\pi/6$, $\pi/6$. The existence of certain multiplier functions is established, these having the character of entire functions and vanishing (of the first order) only at the fixed points for elliptic substitutions. In terms of these simplest functions, all other functions of the group are expressed.

6. Dr. Huntington shows that a class K with a rule of combination $\oplus$ will be equivalent to the class of *positive integers* with the rule of combination $+$, when and only when the following eight postulates are satisfied:

I–IV. The same as postulates I–IV in the paper below, (number 7).

V′. If there is an element E having the property described in postulates VI′, then for every element a there is a natural number m such that $a \leqq mE$.

[Here $a \leqq b$ has the same significance as in V, in the paper cited, while mE indicates the element defined by the following recursion formulæ: $1E = E$, $2E = 1E \oplus E$, $3E = 2E \oplus E$, $\cdots$, $(m+1)E = mE \oplus E$.]

VI′. There is an element E such that $x \oplus y \neq E$ whenever $x \neq E$ and $y \neq E$.

VII, VIII. The same as postulates VII, VIII in the paper cited.

The postulates of this third set, like those of the two earlier sets (of six and of four postulates respectively: *Transactions*, volume 3, page 280), are shown to be consistent, sufficient and independent.

7. Dr. Huntington shows that a class K with a rule of combination $\oplus$ will be equivalent to the class of *positive real quanti-*

ties with the rule of combination $+$, when and only when the following eight postulates are satisfied:

I. $a \oplus b$ belongs to the class whenever a and b belong to the class.

II. If $a \neq b$, then $a \oplus b \neq a$.

III. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, whenever $a \oplus b$, $b \oplus c$, etc., belong to the class.

IV. If $a \neq b$, then there is either an element x such that $a = b \oplus x$, or an element y such that $a \oplus y = b$.

V. (Let the notation $a \prec b$ indicate that an element y exists such that $a \oplus y = b$; and let $a \leqq b$ indicate: $a \prec b$ or $a = b$.) If S is an infinite sequence of elements (a_k), such that

$$a_k \prec a_{k+1},\ a_k \prec c \qquad (k = 1, 2, 3, \cdots)$$

(where c is some fixed element), then there is an element A having the following two properties:

1) $a_k \leqq A$ whenever a_k belongs to S.

2) If $B \prec A$, then there is at least one element of S, say a_r, for which $B \prec a_r$.

VI. For every element a there are two elements x and y such that $x \oplus y = a$.

VII. $a \oplus b = b \oplus a$ whenever a, b, $a \oplus b$, and $b \oplus a$ belong to the class.

VIII. There are two elements x and y in the class, such that $x \neq y$.

The postulates of this new set, like those of the earlier set of six postulates (*Transactions*, volume 3, page 264) are shown to be consistent, sufficient, and independent.

8. Dr. Fite's paper concerns itself with the generalization of some theorems proved by him in the *Transactions* and in the BULLETIN. It considers the invariance of certain powers of operators of groups whose orders are powers of a prime, and also the possibility of certain groups occurring as groups of cogredient isomorphisms.

9. Professor Wilczynski gave a complete geometric interpretation for the covariant C_3 in the case where the flecnode curve intersects every generator in two coincident points, the interpretation given for the general case not being applicable. He further showed how to associate, with every generator of the ruled surface, a certain conic, the derivative conic. This

gives rise to a surface, associated with the ruled surface, upon which are situated a single infinity of conics. A number of theorems were proved in regard to the derivative conic, and also in regard to the developable surface enveloped by the plane of the derivative conic.

10. By employing the same methods as those published in the *American Journal*, volume 25, and the additional property that a bitangent from a point on the directrix to a plane section of a scroll shows the presence of a double point on the nodal curve, Professor Snyder has proved the existence and obtained the equations of over one hundred forms of sextic scrolls having a rectilinear directrix. The list is believed to be complete.

11. If three points are severally on and collectively apolar with a cubic curve in a plane, four linear conditions are imposed on the cubic. Two such 3-points in general impose eight conditions and determine a pencil of cubics, which intersect also in a third 3-point.

In Professor Morley's paper it was shown that the three 3-points are mutually related, so that the 3-points of a plane, projectively considered are in a relation like an involution, say a triplicity. In proof, two cubics of the pencil are taken, one the resultant $(abx)^3$ of the 3-points $(a\xi)^3$ and $(b\xi)^3$, the other a rational cubic, along which a parameter κ is spread. The three triads of parameters where the cubics meet are shown to be mutually related.

For proper 3-points, that is, not on a line, there is the exceptional or 'neutral' case when the resultant explodes; that is, the two given 3-points appear apolar from every point on the plane. When two such 3-points are on a cubic, they are apolar with it, so that the eight conditions cease to be independent.

12. The general surface of singularities in the spherical quadratic complex has been studied by Professor P. F. Smith in the *Transactions*, volume 1, page 381. By using the same methods and supplementing them with the types of cyclides as given by Loria, Mr. Moore has completed the classification of all the types, including those which are annular surfaces, scrolls, developables, cyclides and minimum lines.

14. The results contained in Mr. Ford's paper relate to the function of the complex variable z defined by the series

$$(1) \qquad a(0) + a(1)z + a(2)z^2 + \cdots + a(k)z^k + \cdots,$$

both in the case where the radius of convergence of the series is not equal to zero and where equal to zero. In the first of these cases a theorem is obtained by which the function defined by (1) may be studied when the same function could not be studied by the direct application of any theorem relating to this subject which has appeared heretofore, including those of Le Roy and Lindelöf and the theorem established a short time since by the author in the *Journal de Mathématiques*. In each of these former theorems, in fact, it is required to know certain facts concerning the behavior of the function $a(k)$ in the neighborhood of the point $k = \infty$. If we have at our command, however, only the means of studying $a(k)$ for finite values of k (as is in general the case), the facts just cited become difficult in many cases to determine. But by means of the theorem which Mr. Ford establishes, one may avoid such difficulty, at least in very many cases, since by it the discussion of the function $a(k)$ is referred to that of two other functions $b(k)$ and $c(w_1, w_2, w_3, \cdots, u_q)$, each of which for a given series (1) may usually be chosen in various ways, at least one of which will be such that the conditions demanded will be satisfied. In the second of the cases cited above a theorem is obtained by which one may study the function defined formally by a *divergent* series (1).

15. The point of view adopted in Professor Smith's paper was announced in a preliminary communication made to the Society in December, 1901, entitled: "On the Resolution of Orthogonal Transformations into Involutory Transformations." The present paper completes the discussion of the general question of linear transformation of an invariant quadratic form, obtaining *in explicit form* all such transformations, discusses the resolution into involutory transformations and closes with an application to the case of six variables, completing the theorem given by Dr. Wilson in the *Transactions* concerning the resolution of a collineation into three skew reflections.

16. The correlation between projective and metric parabolic geometry seems unsatisfactory: for first, the method of using the absolute depends on imaginaries and on a limiting process, and secondly, the procedure employed in most books on pure geometry avoids the question entirely and merely states metric

results in projective language. In this note, which has been offered to the *Annals of Mathematics*, Dr. Wilson shows how to pass from the general projective geometry to metric parabolic geometry by means of a fundamental unit conic and Von Staudt's theory of involutions.

17. Mr. Sisam discussed a depiction of the lines of a special linear complex on the points of space, as developed by Anders Wiman, of Lund, and applied it to the study of quintic scrolls. The polar planes of an arbitrary point with respect to a pencil of quadrics intersecting in two conics form a pencil whose axis cuts the line of intersection of the planes of the conics. By letting this axis correspond to the given point, we have the depiction required. A scroll belonging to the complex depicts into a curve. By studying the curves corresponding to quintic scrolls, thirty-eight types were found which have a rectilinear directrix.

18. Dr. Kasner's paper is a contribution to the geometry of ordinary differential equations, that is, of systems of curves. Three topics in particular are discussed: 1° The determination of the linear differential equations which represent isothermal systems. 2° The derivation of new isothermal systems from known systems depending on the consideration of *slope* systems (the slope curves of a given system $y' = f(x, y)$ are represented by $f(x, y) = \text{const.}$). 3° The characterization of an isothermal system in terms of the isogonal trajectories. The result obtained here applies to systems on a surface as well as in a plane, and is as follows: The ∞^2 isogonal trajectories of a given system of ∞^1 curves constitute a linear two-parameter system when, and only when, the given system is isothermal. Incidentally certain general theorems on isogonal trajectories are obtained.

F. N. COLE.

COLUMBIA UNIVERSITY.

THE CASSEL MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held, in connection with the seventy-fifth convention of the Deutsche Naturforscher und Aerzte, at Cassel, September 20–26, 1903. More than two thousand persons attended the convention and were royally entertained by the citizens of Cassel. A reception to visitors, on Sunday evening, marked the formal opening of the convention; on Monday and Friday mornings occurred general meetings in the Stadtpark; and on Thursday morning the regular business meeting was held. The other mornings and afternoons of the week were devoted to the reading of papers before the respective sections, which met in different parts of the city.

The weather was ideal and many excursions were arranged to different parts of the surrounding country, in particular to the wonderful artificial waterfall and to the Emperor's summer castle on Wilhelmshöhe, about five miles west of the city. Visits were also made to the extensive locomotive works situated here. Cassel has a fine picture gallery which was well patronized during the week. Each evening had its own particular entertainment, beginning Monday evening with an opera, seats to which were reserved for all registered visitors. Tuesday evening occurred a banquet and on Friday evening a "Trinkfest" at the expense of the city of Cassel. On Wednesday evening the members of the Mathematical Section arranged to dine together.

An invitation was extended by the faculty and officers of the University of Göttingen to visit the university Friday afternoon and Saturday morning for the purpose of inspecting the equipment, particularly in the departments of mathematics, chemistry, physics, astronomy, and agriculture. About one hundred and thirty persons accepted the invitation.

A committee appointed to consider the question of founding a central mathematical library reported that the firm of B. G. Teubner, in Leipzig, had offered to present a complete set of its publications as a nucleus for the library and that the firm further offered to assist in cataloguing its gift. The offer was gratefully accepted.

The next meeting of the Vereinigung will be held at Heidelberg, August 8–13, 1904, in connection with the third International mathematical congress. Notices of this congress have already been published in the BULLETIN (volume 9, page 565; volume 10, page 94).

At the Cassel meeting the Vereinigung under the chairmanship of Professor Klein held six sessions, at which the following papers were read:

1. Professor SCHEFFERS, Darmstadt: "Sophus Lie."
2. Professor LAMPE, Berlin: "On the life and work of M. Hamburger."
3. Professor FRICKE, Braunschweig: "On new methods and new text-books in England."
4. Dr. GEISSLER, Charlottenburg: "Foundations of non-euclidean geometry subject to conditions on the infinite elements."
5. Dr. BERNSTEIN, Halle: "On abelian 'Körper' in an imaginary fundamental domain."
6. Dr. BLUMENTHAL, Göttingen: "On abelian functions and modular functions of several variables."
7. Professor BOLTZMANN, Vienna: "On an extension of the Lagrangean equations for the case of non-holonome coördinates."
8. Professor BURKHARDT, Zürich: "Report on the development of the oscillatory functions."
9. Dr. HAMEL, Carlsruhe: "On the meaning of the Lagrangean 'Transitivitätsgleichungen' in mechanics."
10. Professor HEFFTER, Bonn: "On systems of teaching geometry, particularly in case of analytic treatment."
11. Dr. LIEBMANN, Leipzig: "A general proof of the Minding theorem on the stability of closed convex surfaces."
12. Professor MASCHKE, Chicago: "On invariants of quadratic differential forms."
13. Professor MEHMKE, Stuttgart: "On graphic kinematics and dynamics."
14. Professor MEYER, Koenigsberg: "On a fundamental equation of surface theory and mechanics."
15. Professor SCHOUTE, Groningen: "Considerations on the content of an n-dimensional prismoid."
16. Professor STÄCKEL, Kiel: "Report on the mechanics of multiple manifolds."
17. Professor SCHOENFLIES, Koenigsberg: "On analysis situs."

18. Professor JUEL, Copenhagen: "Volumes of pyramids."

19. Professor WIEN, Würzburg: "On differential equations of electrodynamics of moving bodies."

20. Professor HILBERT, Göttingen: "Mechanics of continua."

21. Professor MINKOWSKI, Göttingen: "On capillarity."

22. Dr. WELLSTEIN, Giessen: "Fundamental deductions from the arithmetic theory of algebraic functions of a single variable."

23. Professor WIENER, Darmstadt: "The problem of the normal to a conic."

24. Professor PRANDTL, Hanover: "On uniform symbolism in vector analysis."

25. Professor WIENER, Darmstadt: "Exhibition of mathematical models."

Professor Maschke's paper has already appeared in the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY. The first two papers of the programme were biographical rather than mathematical; no abstracts of these are offered. Abstracts of the other papers, so far as obtainable, are as follows:

3. Professor Fricke outlined what Professor Perry is endeavoring to do in England and compared the Perry movement in England with a parallel movement in the German universities. He also exhibited copies of English text-books of elementary mathematics. An extended report on the subject will appear in the *Jahresbericht* of the Vereinigung.

4. Dr. Geissler extended the archimedean continuity of the finite to a general continuity of the infinite. This theory is not in part discontinuous but is in general continuous. Similarily, the hypereuclidean geometry which he is about to publish is not anti-euclidean or non-euclidean, but is in the same domain as euclidean geometry. Among the new concepts in this book are the following: There exist several parallels through one point which cut a second straight line in different points at infinity; the theorem on the sum of angles is, in many cases, independent of the axiom of parallels.

5. Dr. Bernstein bases his work on theorems of Kronecker, Weber, and Hilbert. He gives the necessary and sufficient conditions that a "Primideal" of the fundamental system (Grundkörper) shall lie in a particular class of the same. He gives further, on the ground of empirical induction, an exact

criterion in the case of a "Primideal" of the fundamental system in the corresponding "Klassenkörper."

6. Dr. Blumenthal showed that as the Jacobian modular function may be represented as the quotient of null values of ϑ-functions of a single variable, so the modular function of several variables (see *Mathematische Annalen*, volume 56) may be expressed by means of quotients of null values of ϑ-functions of several variables. This theorem, which originated with Professor Hilbert, rests on the theory of algebraic "Zahlkörper" and can be applied for any arbitrary number of variables. The paper will appear in the *Jahresbericht*.

7. Professor Boltzmann showed how the correction terms which the Lagrangean equations require are to be computed in case the generalized coördinates p which are used are not holonome, *i. e.*, where a non-integrable relation of the form

$$dx = \sum_{h} dp_h f_h(p_1, p_2, \cdots, p_n)$$

exists between the coördinates p and the rectangular coördinates x. He proposes, further, whenever the law according to which the forces act, as well as the determining equations, involves the time explicitly, that the given mechanical system shall be called a "skleronom" system. Similarly the generalized coördinates shall be called skleronom if they are connected with the rectangular coördinates by equations which do not contain the time explicitly. The opposite case he calls "rheonom."

8. Professor Burkhardt has collected and arranged in chronological order material on the development of oscillatory functions. Already three parts, comprising 768 pages, have appeared in volume 10 of the "*Jahresbericht*," in the years 1901, 1902, 1903, respectively. The fourth and last part is now in press. Professor Burkhardt's paper was a brief report on the contents of this fourth part, emphasizing some of the more recent and important developments in the subject.

9. Dr. Hamel proposed the problem: A generalization of the Lagrangean equations of mechanics when the position of the system is represented by the coördinates $q_1, \cdots, q_n$, and the

velocity of the system is represented, not by the derivatives $q_1', \cdots, q_n'$ with respect to time, but by linear, otherwise arbitrary (hence non-holonom), relations

$$\omega_\lambda = \sum_{\lambda=1}^{n} \pi_{\lambda\rho} q_\rho' \cdots \tag{1}$$

between these derivatives. In order to solve the problem it is necessary to deduce the "Transitivitätsgleichungen" which determine the variations $\delta\omega_\lambda$ in terms of the derivatives of the actual displacements

$$\delta D_\lambda = \sum_{\lambda=1}^{n} \pi_{\lambda\rho} \delta q_\rho \cdots, \tag{2}$$

corresponding to equation (1).

Dr. Hamel illustrates the advantage of the Lagrange-Euler equations of mechanics in handling non-holonom equations of condition, and indicates briefly their relation to Lie's theory of transformation groups. His paper will appear in full in the *Zeitschrift für Mathematik und Physik*.

10. Professor Heffter's paper is in abstract as follows: If "point," "plane," and "straight line" are taken as elements of space, and "incidence," "parallelism," and "orthogonality" as the three elementary relations, then geometry separates into the three divisions: projective, affine, conform, corresponding to the three similarly named transformation groups. The projective geometry depends on incidence; the affine on incidence and parallelism; the conform on incidence, parallelism, and orthogonality. In the analytic treatment the notion of a characteristic absolute invariant of transformation groups or geometry will be taken as fundamental. As an example, in plane projective geometry we have the anharmonic ratio of point or line pairs; for the affine geometry, the distance ratio of a straight line from point pairs; for conform geometry, the distance ratio of a point with respect to straight line pairs. The natural coördinates for each division of geometry will be chosen with the help of a particular characteristic absolute invariant.

11. Dr. Liebmann corrected an error in a previous paper and showed that the general spherical and closed convex surface

is non-deformable only under initial assumptions on the nature of the deformation.

13. Professor Mehmke utilizes the hodograph (Möbius-Hamilton) and shows how a graphic representation for n-dimensional space may be developed. For motion in space of more than three dimensions, a generalization of descriptive geometry is needed, which can be found in Professor Schoute's book on Mehrdimensionale Geometrie. As the acceleration is found from the velocity, so the "hyper-acceleration" (acceleration of 2d order or velocity of 3d order) can be found from the ordinary velocity and acceleration. In general the velocity of $(n + 1)$th order can be found from those of the first and nth orders.

The construction furnishes a general graphic solution of the problem: To determine the path of a point when, instead of the initial position and initial velocity, the accelerating force for each point is given, a problem which has been solved only for motion in a plane under particular restrictions. The paper will appear in full in the *Zeitschrift für Mathematik und Physik*.

14. The well known transformations that Lagrange has employed on his equations in dynamics, in which, according to assumptions in case of a function of forces, a material point is constrained to lie on a given surface, is from the standpoint of surface theory nothing other than a systematic introduction of Gauss's first surface form in the equation of motion. The question of a corresponding relation in the case of Gauss's second surface form has led Professor Meyer, without any restriction in respect to the force, to a simple law existing between the Huyghen's normal force (centrifugal or centripetal) belonging to the path curve and the tangential force belonging to the surface. The deduction for the radius of curvature of a curve on the surface is, in this case, extremely simple. An application of the law gives a general mechanical definition of geodesic lines as well as of asymptotic lines of a surface.

Different forms of the law are derivable by direct application of simple principles of mechanics and geometry. The law is also applicable to n-dimensional space.

15. Professor Schoute deduces new expressions for the volume of an n-dimensional prismoid, *e. g.*, for $n = 5$,

$$I = \frac{h}{18}\left[8D_i + 5(D_{i-h\sqrt{5}} + D_{i+h\sqrt{5}})\right],$$

and incidentally finds the following theorem:

If $1, 2, \cdots, n-1, n$ be the roots of the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0,$$

the equation

$$\frac{a_0}{n+2}x^n + \frac{a_1}{n+1}x^{n-1} + \cdots + \frac{a_{n-1}}{3}x + \frac{a_n}{2} = 0$$

admits only one pair of real roots for even n, and only one real root for odd n; the roots of the pair are equal to n, and the unique root lies between n and $n+1$.

As Professor Schoute states, it is not this theorem itself, but its hyperdimensional origin that perhaps may be of interest.

16. Professor Stäckel reviews the notion of multiple manifolds of mechanics and takes up the following problems: The case of motion of a point in a multiple manifold in which the differential equations of the motion can be integrated by quadrature; the problem of the planets in non-euclidean space; finally the analytic equivalence of problems of dynamics and the application of Lie's theory of transformation groups to dynamics. He considers the relation between multiple manifolds of mechanics and numerous chapters of higher analysis and geometry on the one side and certain realms of mathematical physics on the other. In the latter connection the investigations of Gibbs are particularly considered.

17. The paradoxes of the "Mengenlehre" having completely upset the notions ordinarily connected with the words "curve," "surface," etc., it is the problem of the theory of "Punktmengen" to establish, by suitable definition, those conceptions with which analysis situs operates. These definitions are, moreover, to be such that they shall remain invariant under all "one-to-one" continuous transformations of the plane; for under these transformations all the theorems of analysis situs remain invariant. Professor Schoenflies shows how this can be easily established in case of the ordinary invariants of the group, namely, the dimension and relation in the realm of Punktmeng-

en. The classification of all plane closely connected Punktmengen shows that the simple curve segment and the simple closed curve is the simplest type; from these the "one-to-one" continuous representation on the circle segment and the circle is established. This method is applicable in space and here also leads to the theorems that are used in analysis situs, and of these the first is the theorem on curves by Jordan.

18. Professor Juel showed, by diagram, how a special four-sided pyramid can be cut up and the parts again put together in the form of a cube. He called attention to a note by himself on this problem published in *Berichten der Kgl. Gesellschaft der Wissenschaften zu Kopenhagen*, 1903, under the title "Egalité par addition de quelques polyèdres." This same problem has been handled by Mr. Hill in the *Proceedings of the London Mathematical Society*, volume 27, but in a slightly different way.

19. Of the differential equations for bodies in motion the system of Lorenz is the only one that has stood the test throughout. These equations have been integrated with exactness only for the case of point charges moving with constant velocity. In Professor Wien's paper the integration has been completed for the case where, instead of stationary charges under known conditions, the generalized case of motion with constant velocity occurs. If for example

$$F\frac{(ct-l)}{r}, \quad r^2 = x^2 + y^2 + z^2$$

is the general integral for rest, when F is any arbitrary function, c the velocity of light, r the distance from a point O, then the integral for motion is

$$F\left(\frac{ckt - vx/k - r}{r}\right),$$

where

$$r^2 = x^2/k^2 + y^2 + z^2, \qquad k^2 = 1 - v^2/c^2$$

One can use this to derive the dependence of the radiation of an electromagnetic vibration on the motion itself. For stationary vibrations the theory has been given by Herz. For

the variable vibration an increase of the radiating energy occurs because of the motion, and in fact, is greater in case of transversal than of longitudinal vibration. For the velocity of light the radiating energy in unit time is infinite.

In case of a transversal vibration the system is such that for infinitely slow vibrations it does not give the same field as a moving electrified double point, according to the general theory. On the contrary there exists a field containing lines of force that vanishes with the vibrations, not with the radiations.

21. A report of Professor Minkowski's paper was to appear in *Nature*.

23. The problem of drawing the four normals to the conic k from any point n which is chosen at random in the plane of its conic was solved by Apollonius. Professor Wiener has treated this problem from the standpoint of modern mathematics, in particular using the theory of binary and ternary forms and has found a purely binary solution.

If we consider the point n as the center of a point involution on the plane k and desire to find, from the double points of this involution, the mixed polar system of the fourth order, the required polars are given by an involution of point pairs, or more exactly, by a set of polar systems of second order; and of these polar systems of the set we find the common apolar system of second order. The mixed polars of the fundamental points of this polar system to the binary polar system of order six is that polar system of fourth order whose fundamental points form the group of the required points. The solution depends then not really on the solution of an equation of the fourth degree but upon the determination of the polar system of the fourth order.

24. Professor Prandtl discusses the methods of representing vectors. He concludes that it is impracticable to use one alphabet (German) for vector quantities and another alphabet (Latin) for absolute values. He proposes that in script underscoring be employed, and in printing, heavy type. The operation symbols $=$, $+$, $-$, can then be employed as in algebra. For multiplication with a scalar the ordinary method of juxtaposition can be employed, as $u = gt$. The product of two vectors may be represented in accordance with Gibbs as follows: $a \cdot b$ means the direct product; $a \times b$ the skew product.

If for pedagogical reasons a particular representation of an operation is desired, a double underscoring, or in case of multiplication parentheses can be employed.

25. In his second paper Professor Wiener explained the construction of new models of surfaces of the second order. Two thread models, the hyperboloid of revolution and the hyperbolic paraboloid, are so constructed that their forms may be changed without changing the length of the threads. Models of the same surfaces are made of light rods fastened together by a ball and socket device. Each rod is joined to other rods at three different points. The forms of these models can be changed. In the case of the hyperboloid of one sheet, the limiting forms, the ellipse and the hyperbola, are exhibited in an extremely neat form.

I wish to thank those who have so generously aided me, by by the loan of papers and otherwise, in the preparation of this report.

R. E. WILSON.

GÖTTINGEN,
November, 1903.

ON A TEST FOR NON-UNIFORM CONVERGENCE.

BY DR. W. H. YOUNG.

(Read before the American Mathematical Society, August 31, 1903.)

1. A PAPER of Cayley's entitled "Note on Uniform Convergence," appeared in 1893 in the *Proceedings of the Royal Society of Edinburgh*. It was reprinted in volume 13 (1897) of Cayley's Collected Works. The only reference to the paper which to my knowledge exists is a remark by Pringsheim in his article in Encyclopädie II A 1, page 34, to the effect that the objections made by Cayley in the paper in question to the usual definition of non-uniform convergence appear to be due to a misconception.

It is clear indeed from the discussion of the definition with which the note commences that Cayley had remained in what may be supposed to have been Stokes's order of ideas at the time of writing his * classical memoir on the subject in 1847,

* Stokes, *Cambr. Math. Soc. Trans.*, vol. 7, p. 533 (1847).

and that Cayley was in 1893 still ignorant of the interesting and valuable investigations made by Darboux,* Du Bois Reymond † and others as long before as 1875. The only points of non-uniform convergence known to him were those for which the sum of the series is discontinuous; and as Cayley lived in close proximity with Stokes and was in constant contact with him, the question is raised as to whether the same might not be said of the latter mathematician. Be that as it may, it is a curious fact that in 1897, the very year of the reprint of Cayley's paper, the problem of the complete distribution of the points of non-uniform convergence of a series of continuous functions whose sum is continuous was finally solved by Osgood,‡ who showed that these points may be dense everywhere.

The early part of Cayley's note is quite interesting and instructive reading to the student who takes it in a critical spirit, but it is not our intention to enlarge upon it here. At the conclusion of the paper Cayley proceeds to formulate a condition for the existence of a point of non-uniform convergence; a condition which is, however, as it stands, inadmissible even for points of the Stokes type.§ In what follows we emend Cayley's test, and show how a modified form of it is applicable to all points of non-uniform convergence, while it at the same time serves to discriminate between the Stokes points § and the other points of non-uniform convergence.

2. Let $u_1(x), u_2(x), \cdots$ be continuous functions of x in a given segment, and let their sum be convergent at every point of the segment and be denoted by $f(x)$. Cayley deduces from the two equations

$$f(x) = u_1(x) + u_2(x) + \cdots, \quad f(a) = u_1(a) + u_2(a) + \cdots,$$

the equation

$$f(x) - f(a) = (x - a)\{v_1(x) + v_2(x) + \cdots\}, \tag{1}$$

where

$$v_n(x) = \frac{u_n(x) - u_n(a)}{x - a}; \tag{2}$$

and from this he derives his criterion, viz., that the series

* Darboux, *Ann. de l'Ecole Normale* (2), 4, p. 77 (1875).
† Du Bois Reymond, *Abh. d. Münch. Ak.*, vol. 12, 1, p. 120.
‡ Osgood, *Amer. Jour.*, 19, p. 155 (1897, presented Aug. 31, 1896).
§ That is, a point at which the sum of a convergent series of continuous functions is discontinuous.

$\sum_1^\infty v_n(x)$ must have "a sum indefinitely large for $x-a$ indefinitely small, or say a sum $= N/(x-a)$."

What Cayley has here apparently overlooked is that the sum of the series may have discontinuities not only of the first but also of the second kind;* if a is a discontinuity of the second kind for $f(x)$, it is clear that $x-a$ may become indefinitely small without $\sum_1^\infty v_n(x)$ becoming indefinitely large, the left hand side of (1) having, without further specification of x, no definite limit.

3. Take for instance the series

$$\sum_1^\infty u_n(x) = \frac{y_1+a}{y_1^2+1} + \left\{\frac{2y_2+a}{2!y_2^2+1} - \frac{y_1+a}{y_1^2+1}\right\} + \cdots + \left\{\frac{ny_n+a}{n!y_n^2+1} - \frac{(n-1)y_{n-1}+a}{(n-1)!y_{n-1}^2+1}\right\} + \cdots,$$

where

$$y_n = x(x-1)(x-\tfrac{1}{2})(x-\tfrac{1}{3})\cdots(x-1/n).$$

At the origin

$$\sum_1^\infty u_n(0) = a,$$

while at any other point x, the sum of the first n terms, say $S_n(x)$, is

$$S_n(x) = \frac{ny_n+a}{n!y_n^2+1},$$

so that $S_n(1/k) = a$ for all values of $n \geq k$, and therefore

$$\lim_{n=\infty} S_n(1/k) = a,$$

while everywhere else

$$\lim_{n=\infty} S_n(x) = 0.$$

We have therefore $f(x) = 0$ at a generic point, but $f(x) = a$ at the origin or at any point $1/k$.

* A discontinuity of the first kind of a function $f(x)$ is a point such that, as x approaches it from the left (or from the right), we obtain a definite limit for $f(x)$. If this is not the case the discontinuity is said to be of the second kind.

Each point $1/k$ is a discontinuity of the first kind, while the origin is a discontinuity of the second kind, all these points being points of non-uniform convergence of the Stokes type.

If we apply Cayley's test to this example we have to consider

$$\sum_1^\infty v_n(x) = \frac{1}{x}\left\{\frac{y_1+a}{y_1^2+1} - a\right\} + \frac{1}{x}\left\{\frac{2y_2+a}{2!y_2^2+1} - \frac{y_1+a}{y_1^2+1}\right\} + \cdots.$$

At the origin

$$\sum_1^\infty v_n(0) = -1 + \{1+1\} - \{1/2!+1\} + \{1/3!+1/2!\} + \cdots,$$

so that the series is convergent there and has the value 0.

At any other point x the sum of the first n terms is $(S_n - a)/x$ and the residue R_n/x. Thus the series is convergent up to and including the origin, having at any generic point x the value $-a/x$, while it vanishes at the origin and at all the points $1/k$. The behavior of the test series along a *generic* sequence is accordingly that suggested by Cayley as characteristic, but along some sequences the function has no determinate limit, and along the sequence $x = 1/k$ it is zero everywhere.

4. It will be preferable to write the equation (1) in the following form:

$$f(x) - f(a) = (x-a)\sum_1^m v_n(x) + R_m(x) - R_m(a). \qquad (3)$$

First let $f(x)$ be discontinuous at $x = a$, as in Cayley's case. Then there is certainly one sequence of points, having a as limit,

$$y_1, \quad y_2, \cdots,$$

passing along which $|f(y_i) - f(a)|$ remains always above some positive limit b. At every point y_i of such a sequence we can, since the series is convergent there, find an integer m_i such that for all values of $m \geq m_i$, $|R_m(y_i) - R_m(a)|$ is smaller than any assigned small positive quantity ϵ.

Thus

$$|(y_i - a)\sum_1^m v_n(y_i)| > b - \epsilon. \qquad (4)$$

Passing therefore along such a sequence $\sum_1^\infty v_n(y_i)$ tends to become infinite of at least the first order in $(y_i - a)$.

5. Next let us assume that $f(x)$ is continuous at a. Then we may confine our attention to points x so near a that the left hand side of the equation (3) remains less than any assigned small quantity ϵ. Also given any sequence in the interval so determined

$$y_1, \quad y_2, \cdots$$

having a as limit, we can assign a corresponding series of integers

$$m_1', \quad m_2', \ldots$$

such that, for all values of $m \geq m_i'$,

$$|R_m(y_i) - R_m(a)| < \epsilon.$$

It then follows from (3) that

$$|(y_i - a)\sum_1^m v_n(y_i)| < 2\epsilon,$$

for all values of $m \geq m_i'$, and therefore

$$|(y_i - a)\sum_1^\infty v_n(y_i)| \leq 2\epsilon.$$

This last inequality shows that no specification of the sequence y_i can render Cayley's criterion applicable to a point of non-uniform convergence other than a Stokes point.

On the other hand, since a is a point of non-uniform convergence, we can, if we choose the sequence y_i suitably, determine a series of integers

$$m_1, m_2, \cdots$$

(having of course no finite limit since $f(x)$ and therefore, for all values of m, $R_m(x)$ is continuous), such that

$$|R_{m_i}(y_i) - R_{m_i}(a)| > 2\epsilon,$$

if ϵ has been previously chosen sufficiently small.

It then follows from (3) that

$$\left|(y_i - a)\sum_1^{m_i} v_n(y_i)\right| > \epsilon. \tag{5}$$

Comparing this with (4) we see that (5) is a *necessary* condition for a to be a point of non-uniform convergence, whether of the Stokes type or not.

6. Take, for instance, the series

$$\sum_1^\infty u_n = \left\{\frac{x}{(x+1)^2} - \frac{2^5x}{(2^3x+1)^2}\right\} + \left\{\frac{2^5x}{(2^3x+1)^2} - \frac{3^5x}{(3^3x+1)^3}\right\}$$
$$+ \cdots + \left\{\frac{n^5x}{(n^3x+1)^2} - \frac{(n+1)^5x}{[(n+1)^3x+1]}\right\} + \cdots,$$

which is non-uniformly convergent at the origin with infinite measure.

Here

$$S_m(x) = \frac{x}{(x+1)^2} - \frac{m^5x}{(m^3x+1)^2}, \quad \sum_1^m v_n(x) = \frac{1}{(x+1)^2} - \frac{m^5}{(m^3x+1)^2},$$

$$R_m(x) = \frac{m^5x}{(m^3x+1)^2}, \qquad \sum_m^\infty v_n(x) = \frac{m^5}{(m^3x+1)^2},$$

$$f(x) = \frac{x}{(x+1)^2}, \qquad \sum_1^\infty v_n(x) = \frac{1}{(x+1)^2}.$$

The series $\sum_1^\infty v_n(x)$ converges non-uniformly with infinite measure in any open interval having the origin as left hand end-point, and diverges at the origin. In this open interval $\sum_1^\infty v_n$ is always less than 1, but $\sum_1^{m_i} v_n(y_i)$ increases without limit for the sequence $y_i = 1/i^3$, when $m_i = i$, and tends to become infinite of the order $5/3$.

7. Conversely if there be at least one sequence

$$y_1, y_2, \cdots$$

having a as limit, passing along which

$$(y_i - a)\sum_1^{m_i} v_n(y_i)$$

does not decrease without limit, then along this sequence or along some partial sequence chosen from it, $|R_m(y_i) - R_{m_i}(a)|$ will either become as small as we please, or else will remain above some finite limit. In the latter case we have the direct condition for non-uniform convergence. In the former case, on the other hand, it follows from (3) that $|f(y_i) - f(a)|$ does not decrease without limit, and $f(x)$ is therefore discontinuous at a which is therefore a Stokes point. Thus we have shown that *the necessary and sufficient condition for the non-uniform convergence of the series* $\sum_1^\infty u_n(x)$ *of continuous functions at the point* $x = a$, *is that we should be able to assign a sequence of points* $y_1, y_2, \cdots$ *having* a *as a limit, and a corresponding series of integers* $m_1, m_2, \cdots$, *such that* $\sum_1^{m_i} v_n(y_i)$ *tends to become infinite of at least the first order in* $(y_i - a)$, *where*

$$v_n(x) = \frac{u_n(x) - u_n(a)}{x - a}.$$

Further if a be a Stokes point we may for m_i in the above test write ∞, or, if we prefer it, m where $m \geq m_i$, while in the contrary case this is not allowable, so that we have a criterion which differentiates the Stokes points from the others.

8. A form of this criterion, not always applicable but useful in a number of special cases, is obtained by observing that, since $u_n(x)$ is a continuous function we can, *if it is also differentiable*, determine a point y, between a and x, so that

$$v_n(x) = u_n'(y)$$

and *

$$(x - a)\sum_1^{m_i} v_n(x) = \frac{x - a}{y - a}\left\{(y - a)\sum_1^{m_i} u_n'(y)\right\}.$$

Hence it is a sufficient criterion in this case for the non-uniform convergence at a *of* $\sum_1^\infty u_n(x)$ *if* $\sum_1^{m_i} u_n'(y)$ *tends to become infinite of at least the first order in* $(y - a)$, *as* y *approaches the point* a *along any sequence having* a *as limit.*

* The y in the following equation is not in general the same as the y in the preceding equation. In the following equation y depends not only on x, but also on m.

This criterion is not necessary since it is conceivable that $\frac{x-a}{y-a}$ might itself tend to become infinite. In this case, however, since $(y-a)\frac{x-a}{y-a}$ becomes as small as we please, $\frac{x-a}{y-a}$ must become infinite of an order less than 1 in $(y-a)$. Thus $\sum_1^{m_1} u_n'(y)$ must certainly become infinite though now of an order less than 1 in $(y-a)$.

We see then that it is a necessary condition that $\sum_1^{m_1} u_n'(y)$ should tend to become infinite, but if the order of infinity in $(y-a)$ is less than 1, a further test is necessary, *e. g.*, that of §6.

9. In the case of a series involving the sign of integration, or when the differential coefficient $u_n'(x)$ is a function of standard form, this test is at once applicable. Thus for instance the series for which

$$u_1 = \int \cos\frac{x\pi}{x^3+1}\,dx,$$

$$u_n = \int \left\{ n\left(\cos\frac{nx\pi}{n^3x^3+1} - 1\right) - (n-1)\left(\cos\frac{(n-1)x\pi}{(n-1)^3x^3+1} - 1\right)\right\} dx,$$

is at once perceived to be non-uniformly convergent at the origin, since the series

$$\cos\frac{x\pi}{x^3+1} + \left\{2\left(\cos\frac{2x\pi}{2^3x^3+1} - 1\right) - \left(\cos\frac{x\pi}{x^3+1} - 1\right)\right\} + \cdots$$

is a stock example of a non-uniformly convergent series whose measure of convergence at the origin is infinite of the first order.

PETERHOUSE, CAMBRIDGE,
May, 1903.

ON THE CONDITION THAT A POINT TRANSFORMATION OF THE PLANE BE A PROJECTIVE TRANSFORMATION.

BY MR. ELIJAH SWIFT.

(Read before the American Mathematical Society, October 31, 1903.)

1. WHILE it is well known that all projective transformations are collineations, the converse has, I believe, never been proved in all its generality. Möbius, in his "Barycentrischer Calcul" proves by means of his net of lines that if we start from any four independent * fixed points in the plane, we can either reach any point of the plane, by means of constructions with a ruler alone, or else come as near it as we please. Thence he infers that there cannot be more than one collineation which carries four given independent points into four other given positions. Since we know that there exists a projective transformation which carries over four arbitrarily given independent points into four other arbitrarily given independent points, we infer that every collineation is a projective transformation. In this reasoning, however, Möbius clearly assumes that he is dealing only with transformations which are in general one-to-one and continuous. There are, however, points in the two planes (the points on the vanishing lines) where the transformation is, strictly speaking, not defined. Thus two questions present themselves:

(1) Is it necessary to require that the collineation be continuous in order that Möbius's theorem be true?

(2) Throughout what part of the plane may we leave the transformation undefined?

It is my object in this paper to prove the following theorem, which answers the first question, and goes a long way toward answering the second.

Suppose we have a one-to-one correspondence between the points of two plane point sets S and S', each of which has an interior point, such that any three collinear points in either set have collinear images. Then the transformation of S into S' is a projective transformation.

* By four independent points in a plane we understand four points no three of which lie on a straight line.

2. It should be clearly understood that the points referred to in this theorem are actual geometric points, neither imaginary points nor points at infinity being considered. There is, however, no objection to our using the term "points at infinity" in the course of the proof to avoid long circumlocutions. In fact we shall find it convenient for this purpose to use the term "projective plane," which plane we understand to contain not merely the actual geometric points, but also an infinite number of points at infinity regarded as lying on a line according to the ordinary conventions of projective geometry.

I shall denote the images of points, etc., by primes, thus P' is the image of P, etc. I shall call the first plane I, the second II.

3. Before passing to the proof of the general theorem, it will be convenient to prove two special cases of it. The first is this:—

CASE A. *Suppose we have a one-to-one correspondence of the points of two planes, such that to every finite point of one corresponds one and only one finite point of the other, and, moreover, the images of collinear points are collinear. Then the transformation is projective.*

Before we can speak of a line L having a line L' as its image, we must make sure that not only do all the points on L correspond to points on a certain line L', but that they correspond to *all* of these points. This, however, is obvious when we recall that all the points on L' have images on L.

In the first place if two lines meet in I their images must meet in II, since their point of intersection has a finite image, and accordingly *the images of parallel lines are parallel.*

If we have three collinear points A, B, and C, such that $AB = BC$, then $A'B' = B'C'$. For if we draw two pairs of parallels through A and C we have a parallelogram, and the point B is determined by the intersection of its diagonals. In II we have a parallelogram with A' and C' for two opposite vertices and B' for the intersection of its diagonals. Therefore $A'B' = B'C'$, and *the image of a point midway between two points is midway between their images.*

The necessary and sufficient condition that six points lie on a conic is that the intersections of opposite sides of the hexagon formed by joining them should lie on a line. Since this condition involves nothing but collinearity of points, if it is true of one set of six points it is true of their images. Take

any conic in one plane, select any five points on it, and consider the conic through their images. If we take a sixth point and let it describe the first conic, its image must always lie on the second conic. Therefore every point on the first conic has an image on the second, and, similarly every point on the second has an image on the first. Therefore one conic is the image of the other, and *the image of a conic is a conic.*

Since the center of a conic bisects every chord through it, its image possesses the same property and is, therefore, the center of the transformed conic, and *the image of a central conic is a central conic.* Since every line through the center of an ellipse cuts the ellipse, but this is not the case for the hyperbola, *an ellipse goes into an ellipse.*

If we have two points A and B on opposite sides of a straight line L, A' and B' lie on opposite sides of L'. For if they did not we could pass an ellipse through A' and B' which did not cut L'. Its image would be an ellipse through A and B not cutting L, which is impossible. *If, therefore, a point B lies between two other points A and C, B' lies between A' and C',* since A and C lie on opposite sides of any line through B.

If we have a set of collinear points equally spaced, their images are equally spaced and follow each other in the same order. For if L, M, N are any three points, L', M', N' are in the same order, for if L and N are on opposite sides of M, L' and N' are on opposite sides of M'. If A, B, C are three consecutive points, since $AB = BC$, $A'B' = B'C'$.

Now take any line in I and an origin O on it. Let P and A be any two points on the line, such that OP and OA are commensurable. Take the common measure of OP and OA and lay it off along OA. Suppose it is contained in OA n times and in OP m times. By laying off this common measure, we obtain a set of points equally spaced on OA. Their images must be a set equally spaced on $O'A'$. There must be n of these spaces in $O'A'$ and m in $O'P'$. Accordingly $O'P' : O'A' = m : n = OP : OA$.

Now suppose that OP and OA are incommensurable. Using as a measure of OP a sub-multiple of OA, which we ultimately allow to approach zero as its limit, we can use the familiar method to show that $O'P' : O'A' = OP : OA$, for we know that if P lies between the points reached by applying this measure k times and $k + 1$ times respectively, P' will lie between their images. *Hence distances along a fixed line from a given point are altered in a fixed ratio.*

Now suppose in the first plane we have a pair of rectangular axes with origin O, and a point P whose coördinates are (x, y). Consider their images in the second plane. We shall have a pair of axes, in general oblique, and a point P' referred to them. Denoting by A and B the orthogonal projection of P on OX and OY respectively, we have $OA = x$, $OB = y$. The image of the rectangle $OAPB$ is a parallelogram $O'A'P'B'$, and the coördinates $(\bar{x}, \bar{y})$ of P' are $\bar{x} = O'A'$, $\bar{y} = O'B'$. Then if l and m are the ratios by which distances are altered along OA and OB respectively, $\bar{x} = lx$, $\bar{y} = my$. If P' be referred to any pair of rectangular axes in the second plane, we have, by a transformation of coördinates,

$$x' = a_1x + b_1y + c_1,\ y' = a_2x + b_2y + c_2.$$

And this is a projective transformation. Thus the theorem A is proved.

4. Another special case is the following:

CASE B. *Suppose we have a one-to-one correspondence of all the points of two projective planes, such that to any three collinear points of one, correspond three collinear points of the other. Then the transformation is a projective transformation.*

If every point of I has a finite image, we have practically Case A. For by that case we see that the transformation of the finite points is effected by a projective transformation; and, since the points at infinity may be determined as the vertices of pencils of parallel lines, it is clear that this same projective transformation carries over the points at infinity to their new positions.

Now suppose that the image of the line at infinity in I is a line, L, in II. Apply to II any projective transformation that will throw L to infinity in a third plane III. To every point of I corresponds one and only one point of III. To any point at infinity in I corresponds a point in L in II, and, therefore, a point at infinity in III, and vice versa. Moreover any three collinear points in I have collinear images in II and hence in III, and vice versa. Then we have just such a transformation of I on III as we considered in the first case under Case B. Therefore the transformation of I on III is a projective transformation. Hence, since the inverse of a projective transformation is a projective transformation, and the succession of two projective transformations is a projective transformation, it follows

that the transformation of I on II is projective. Thus theorem B is established.

5. Before proving the general theorem, I shall present a couple of lemmas.

Suppose we have three lines, $\alpha_1, \alpha_2, \alpha_3$ (Fig. 1). Draw any other two lines, β_1, β_2, cutting $\alpha_1, \alpha_2, \alpha_3$ in the points A and B, Q_1 and Q_2, E and F respectively. Let P_1 be the intersection of BQ_1

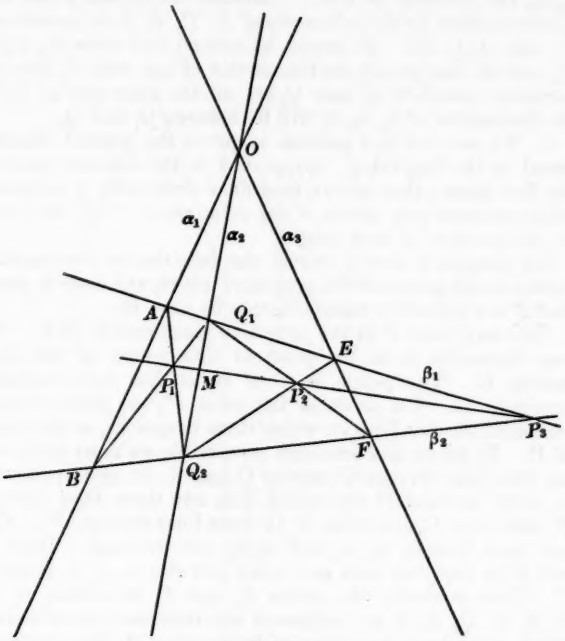

FIG. 1.

and AQ_2, P_2 of EQ_2 and FQ_1, P_3 of β_1 and β_2. *Then a necessary and sufficient condition that* $\alpha_1, \alpha_2, \alpha_3$ *be concurrent is that* P_1, P_2, P_3 *be collinear.* This can be proved very easily by a projection, or is at once evident when we observe that P_1P_3 is the polar of the intersection of α_1, α_2, and P_2P_3 that of the intersection of α_2, α_3 with regard to the degenerate conic consisting of β_1 and β_2.

It is easy to obtain from this a condition for the collinearity of three points. If the points are P_1, P_2, P_3, choose any two points Q_1, Q_2. Let A be the intersection of Q_1P_3 and Q_2P_1, B that of Q_2P_3 and Q_1P_1, E that of Q_1P_2 and Q_2P_2, and F that of Q_2P_3 and Q_1P_2. We have just the same figure as we had before for the three lines, and since the collinearity of P_1, P_2, P_3 was both necessary and sufficient for the concurrence of α_1, α_2, α_3, the converse is true. Therefore *the necessary and sufficient condition for the collinearity of* P_1, P_2, P_3 *is the concurrence of* AB, Q_1Q_2, EF. It should be noticed that since β_2, P_3P_1, β_1, and the line joining the intersection of $\alpha_1\alpha_2$ with P_3 form an harmonic pencil, if Q_1 and Q_2 are on the same side of P_1P_3, the intersection of α_1, α_2, α_3 will lie between Q_1 and Q_2.

6. We are now in a position to prove the general theorem stated at the beginning. Suppose A is the interior point of the first plane; then we can describe a circle with A as center, which contains only points of the point set S. Call this circle C; all points of C have images.

My purpose is first to extend the definition of the transformation to all points of the projective plane, and thus to prove that it is a projective transformation by Case B.

Take any point P in the projective plane outside of C. We may determine it as the point of intersection of two lines cutting C. The points in C of either line have collinear images in II. Let us define the point P', the point of intersection of the two lines on which these images lie, as the image of P. To prove this definition permissible we must show that any two lines through P cutting C lead to the same point P', or, what is evidently equivalent, that any three lines through P, which cut C, determine in II three lines through P'. Call our three lines α_1, α_2, α_3, and apply our first test. Draw β_1 and β_2 so that they meet each other and also α_1, α_2, α_3 inside of C. Then evidently the points P_1 and P_2 lie within C, for A, B, Q_1, Q_2, E, F, are vertices of non-reëntrant quadrilaterals, which must have the points of intersection of their diagonals within C. Then since the points P_1, P_2, P_3 are in C and collinear, their images in II are collinear, and the three lines determined by the images of points on α_1, α_2, α_3 meet in a point P'.

If P is a point of S, it is evident that the image thus defined is the same as the image originally given. For if we take any two points in C collinear with it, their images are collinear with its image.

Now consider any three collinear points P_1, P_2, P_3; we wish to prove that their images, as we have just defined them, are collinear. If the line $P_1P_2P_3$ cuts C, these points are deter-

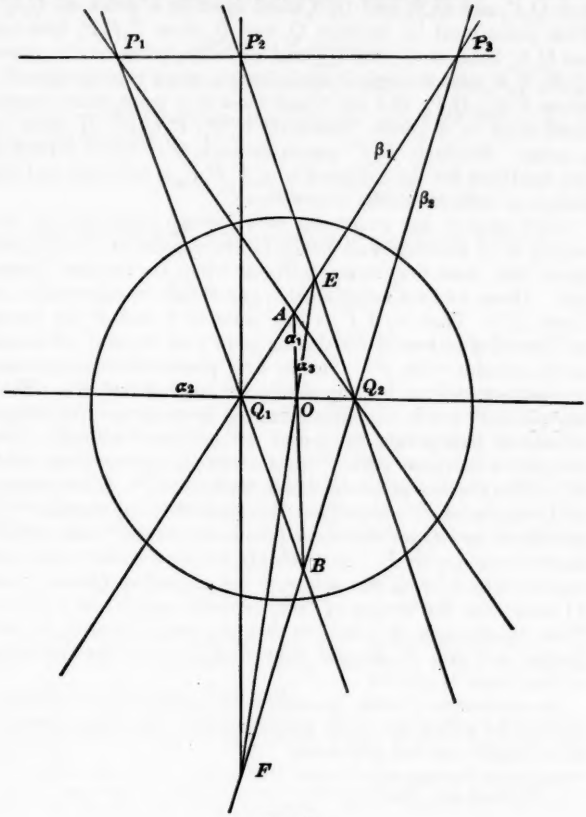

FIG. 2

mined by the intersection of the line $P_1P_2P_3$ and any other lines through them cutting C. Consequently their images must lie on the line determined in II by the images of points on $P_1P_2P_3$.

If $P_1P_2P_3$ does not cut C, choose any two points Q_1 and Q_2 in C and apply our second test. The lines joining the intersections of Q_1P_3 and Q_2P_1 and Q_2P_3 and Q_1P_1, also of Q_1P_3 and Q_2P_2 and Q_2P_3 and Q_1P_2 must meet in a point on Q_1Q_2. This point must lie between Q_1 and Q_2 since $P_1P_2P_3$ does not cut Q_1Q_2 between Q_1 and Q_2, and accordingly lies in C. Since Q_1P_i, Q_2P_i pass through P_i their images must pass through P_i'. Since $P_1'Q_2'$, $Q_1'P_3'$, OA cut C and meet in a point, their images must meet in a point. Similarly $P_1'Q_1'$, $P_3'Q_2'$, $O'B'$ meet in a point. Similarly $E'F'$ passes through O'. Then evidently the condition for the collinearity of $P_1'P_2'P_3'$ is satisfied, and the *images of collinear points are collinear.*

Now since S' has an interior point we can apply all our reasoning to II and define an image of every point of it in I, and show that these images are collinear when the original points are. Have we thus established a one-to-one transformation of I and II? That is, if P is any point of I and P' its image in II, and if in turn P'' is the image in I of P', will P'' necessarily coincide with P? In the first place there is a one-to-one correspondence, by hypothesis, between S and S'. Take any point P not in S. We define P' by means of the images of sets of two points in S, each set collinear with P. Any two points collinear with P' must have images collinear with P''. Take the four points by which we defined P'. Their images in I determine P'' uniquely. But since they are points of S', they have as images the same points we started with, and P'' must coincide with P. Accordingly we have a one-to-one correspondence between the points of the projective planes I and II such that the images of three collinear points are collinear. Then by theorem B it follows that the transformation is projective, and thus the general theorem stated near the beginning of this paper is proved.

In conclusion I wish to express my gratitude to Professor Bôcher for aiding me in the preparation of this paper, by valuable suggestions and criticisms.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

NOTE ON CAUCHY'S INTEGRAL.

BY DR. O. D. KELLOGG.

(Read before the American Mathematical Society, December 28, 1903.)

THE analogy between the formula given by Green for a potential function

$$u(x, y) = \frac{1}{2\pi}\int_C u(s)\frac{\partial G}{\partial n}\,ds \tag{1}$$

and Cauchy's integral representation of a complex function

$$f(z) = \frac{1}{2\pi i}\int_C f(c)\frac{dc}{c - z} \tag{2}$$

has been pointed out;* the direct deduction of one from the other may be of interest.

We start with the case where the curve C is a circle of radius 1. Let $z = x + iy = re^{i\vartheta}$ represent the variable point within the circle; let $c = a + ib = \rho e^{i\sigma}$ represent a parameter point within or on the circle and $c' = e^{i\sigma}/\rho = c/\rho^2$ the reflection of the point c with respect to the circle. Then Green's function for the circle is the real part of $\log [c(z - c')/(z - c)]$, so that if $\Re$ denote "the real part of," the formula (1) may be written

$$u(x, y) = \frac{1}{2\pi}\int_0^{2\pi} u(s)\frac{\partial}{\partial n}\Re \log\frac{c(z - c')}{z - c}\,ds$$

Noting however that the real and imaginary parts of the logarithm are conjugate functions,† we have, if $v(x, y)$ denote the function conjugate to $u(x, y)$,

*See the article in the Encl. d. Math. Wiss: "Analytische Functionen complexer Grössen" (p. 17), by Professor Osgood, to whose suggestion this note is due.

†The fact that the derivatives with respect to the normal of a given curve with a determinate tangent of two conjugate functions G and H are still conjugate functions may be verified as follows. G and H satisfy the equations

$$\partial G/\partial x = \partial H/\partial y, \quad \partial G/\partial y = -\partial H/\partial x. \tag{a}$$

If the direction cosines of the given curve be $\cos\alpha(s)$, $\cos\beta(s)$, then by definition

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial a}\cos\beta(s) + \frac{\partial G}{\partial b}\cos\alpha(s),$$

$$u(x, y) + iv(x, y) = \frac{1}{2\pi}\int_0^{2\pi} u(s)\frac{\partial}{\partial n}\log\frac{c(z-c')}{z-c}\,ds + ic_2,$$

where c_2 is a real constant appearing because a conjugate function is only determined to within an additive constant. Similarly

$$v(x, y) - iu(x, y) = \frac{1}{2\pi}\int_0^{2\pi} v(s)\frac{\partial}{\partial n}\log\frac{c(z-c')}{z-c}\,ds - ic_1,$$

and multiplying this equation by i and adding it to the preceding, we have

$$2[u(x, y) + iv(x, y)] = 2f(z)$$
$$= \frac{1}{2\pi}\int_0^{2\pi} f(c)\frac{\partial}{\partial n}\log\frac{c(z-c')}{z-c}\,ds + c_1 + ic_2.$$

But

$$\frac{\partial}{\partial n}\log\frac{c(z-c')}{z-c} = -\frac{\partial}{\partial\rho}\log\rho e^{i\sigma}\frac{(z-e^{i\sigma}/\rho)}{z-\rho e^{i\sigma}}$$

and in this, after differentiating, we are to set $\rho = 1$. This gives

$$-1 - \frac{2e^{i\sigma}}{z-e^{i\sigma}}.$$

We have therefore

$$2f(z) = -\frac{2}{2\pi}\int_0^{\pi} f(c)\frac{e^{i\sigma}}{z-e^{i\sigma}}\,ds - \frac{1}{2\pi}\int_0^{2\pi} f(c)ds + c_1 + ic_2.$$

To determine $c_1 + ic_2$, put $z = 0$, and note that according to potential theory

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(c)ds.$$

It appears that $c_1 + ic_2$ has this same value, and that the last three terms of the equation fall out, leaving

$$2f(z) = -\frac{2}{2\pi}\int_0^{2\pi} f(c)\frac{e^{i\sigma}}{z-e^{i\sigma}}\,ds,$$

$$\frac{\partial H}{\partial n} = -\frac{\partial H}{\partial a}\cos\beta(s) + \frac{\partial H}{\partial b}\cos\alpha(s).$$

Then $\partial G/\partial n$ and $\partial H/\partial n$ satisfy equations (a) provided the order of differentiation of G and H with respect to a or b and with respect to x or y can be inverted. This is clearly the case for the function in our problem so long as (x, y) keeps *within* the circle.

or, since $e^{i\vartheta}d\vartheta = dc/i$ and $e^{i\vartheta} = c$ on the circumference,

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi} f(c)\frac{dc}{c-z}.$$

The transition from the case of the circle to any region which can be conformally represented on the circle is easy, as Green's function is transformed into Green's function for the new region and $dc/(c-z)$ differs from the corresponding expression for the new region only by a function which disappears upon integration.

PRINCETON, N. J.,
November 20, 1903.

BAUER'S ALGEBRA.

Vorlesungen über Algebra. Von GUSTAV BAUER. Herausgegeben vom Mathematischen Verein München. Leipzig, B. G. Teubner, 1903. vi + 376 pp.

THIS volume was planned in honor of the 80th birthday of Professor Bauer by the Mathematischer Verein of the students of the university and the technical high school of Munich. It presents in fact, not merely in title, lectures as actually given to students in their first or second year at the university, the course extending over two semesters. The preface is by Karl Doehlemann, who saw the book through the press at the request of the Verein.

Treating a wide range of subjects in a strictly elementary manner with many illustrative examples considered in detail, these lectures are certainly very attractive. If one can overlook the lack of rigor in two or three fundamental matters (discussed in detail below), one must regard the volume as one to be specially commended to beginners.

Part I (105 pages), is entitled "General properties of algebraic equations." The usual elementary theorems on complex quantities are given in 12 pages. In the construction of z^n by a series of similar triangles (page 13), n is restricted to positive integers, whereas the series may be continued in the opposite direction to give the negative integral powers. The mere statement that an elementary geometric construction for $z^{1/n}$ is impossible in general would attract the student more if accom-

panied by the remark that it involved the division of an angle into n equal parts. Chapter III deals with the properties of an integral rational function $f(z)$ of a real or complex quantity z. Its rth derivative is defined as the coefficient of h^r in the expansion (by use of the binomial theorem) of $f(z+h)$. There are established the continuity of $f(z)$ for every finite z, the identity of $f'(z)$ with the calculus limit, the theorems on the derivative of a sum, product or power, and Euler's theorem on homogeneous functions.

Chapter IV begins with the theorem that every algebraic equation has a root, numerous references being given. The proof, however, lacks rigor,* as it merely shows that when $f(z) \neq 0$ a complex quantity h can be found such that $|f(z+h)| < |f(z)|$. Chapter V presents in an unusually simple and attractive form the subject of the decomposition of rational fractions, with application to Lagrange's interpolation-formula. The 18 pages of Chapter VI on symmetric functions is elementary in character, Waring's formula not being established. It is perhaps unfortunate to use as an illustration of a rather elaborate process an example (page 54) which the alert student will doubtless solve almost by inspection. Thus if α, β, γ are the roots of $f(x) \equiv x^3 + a_1x^2 + a_2x + a_3 = 0$, we have directly

$$(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) = (-a_1-\gamma)(-a_1-\alpha)(-a_1-\beta)$$
$$= f(-a_1) = a_3 - a_1a_2.$$

Elimination is treated partly in Chapters VII–IX and partly in Chapter XXVIII under determinants. As it stands, § 208 presents the usual incomplete derivation of the resultant of two equations in x by Sylvester's dialytic method. The author apparently does not appreciate the need of proving that the values obtained for the various powers of x are consistent. It would have sufficed to refer to § 210, from which the proof follows.

The statement at the top of page 69 that two equations in two variables have always a definite finite number of common root pairs is subject to the (trivial, to be sure) exception arising when the coefficients of one function are proportional to those of the other. A word on the geometry of the problem would

*As shown by the simple example given by Professor Moritz, "On certain proofs of the fundamental theorem of algebra," *American Mathematical Monthly*, vol. 10 (1903), top of p. 160.

add to the interest of the student. The equality of the degrees of the two resultants (from the elimination of x and y, respectively) would then be expected.

Chapter XI presents the transformations of Tschirnhaus and Jerrard, and discusses the construction of an equation whose roots are specified functions (for example, the differences) of the roots of a given equation.

Part II (pages 106–199) treats of the algebraic solution of equations — Cardan's formula, the irreducible case, Euler's and Lagrange's methods, reciprocal and binomial equations, roots of unity, irreducibility of $(x^p - 1)/(x - 1)$, primitive roots.

Chapter XV is devoted to the Abel-Wantzel theorem that the general equation of degree $\geqq 5$ is algebraically insolvable. There is, however, a painful hiatus in the failure to prove (or notice the need of proof of) Abel's theorem that *every equation which is solvable by radicals can be reduced to a chain of binomial equations (of prime degrees) whose roots are rational functions of the roots of the given equation.*

In the 13 pages of Chapter XVI, the student is given a first view of Galois's theory of algebraic equations. In the opening sections, a factor of an integral function $f(x)$ is called 'rational when its coefficients are rational functions of $f(x)$.' The author should of course say rational functions with integral coefficients. Several pages later is given an extension of the idea of rational factor: 'The coefficients of the equation may be expressible rationally in terms of quantities $e_1, e_2, \cdots$ considered as known. We may then understand by rational factor a factor whose coefficients likewise are formed rationally from $e_1, e_2, \ldots$.' The author is certainly not fortunate here in his introduction of the idea of a domain of rationality. The latter should have an independent definition, for example as the totality of rational functions with integral coefficients of certain quantities $q_1, q_2, \cdots$. When the problem relates to a given algebraic equation, we naturally take as the initial domain of rationality one including all the coefficients, but may agree to admit also quantities either wholly arbitrary or involving irrationalities not appearing in the coefficients. Without this general conception of a domain of rationality, the idea of enlarging the domain by adjunction would be undesirably restricted.

The rudiments of the theory of numbers are given in the 20 pages of Chapter XVII. The two following chapters deal with abelian and cyclotomic equations, with a detailed discus-

sion of the division of the circumference into 5, 7, 13 and 17 equal parts.

Part III (pages 200–350) presents the numerical solution of equations under the topics: limits for real roots; Descartes's rule of signs *; the theorems of Rolle, Fourier, Sturm; methods of approximation due to Newton and Lagrange, and the simple *Regula falsi*; the theorem of Cauchy for the number of imaginary roots within a given region of the complex plane. The method of solution of numerical equations over which the author is most enthusiastic is that of Graeffe, as improved by the astronomer Encke. The principle is the derivation of a second equation whose roots are such high powers of the roots of the given equation that the powers of the smaller roots are, in relation to those of the larger roots, negligible quantities.

Part IV (pages 257–373) is concerned with the theory and application of determinants. The ordinary rule for the multiplication of two determinants of equal order is established very ingeniously by use of linear transformation. However, one step is far from clear (bottom of page 292), as conceivably there is a literal factor of AB which divides the coefficient of each c_{ij} in the expansion of the numerator. Thus, we might have

$$y_k = \frac{c_1 a_1^{n-1} b_1^{n-1}}{a_1^n b_1^n} = \frac{c_1 a_2^{n-1} b_2^{n-1}}{a_1 b_1 a_2^{n-1} b_2^{n-1}}.$$

Application is made to elimination, discriminants, quadratic and bilinear forms.

The volume concludes with two rather long notes on continued fractions and Lagrange's formula for the sum of the nth powers of the roots of a quadratic equation.

Among the misprints not noted in the full page of errata are fz in § 15, line 2; $\alpha^2 - 1$ on page 56, line 18; 2 for r in (3), page 80; $1 \cdot 2 \cdots r$ for $1, 2, \cdots, r$ on page 97, lines 19 and 23, and similarly $x_1 \cdots x_n$ on pages 98, 99 and elsewhere.

The presswork has the usual excellence of Teubner's books. There is a portrait of Professor Bauer as frontispiece.

L. E. DICKSON.

THE UNIVERSITY OF CHICAGO,
November 10, 1903.

* In the form: "An equation with real coefficients has as many positive roots as changes of sign, or fewer by an even number."

SHORTER NOTICES.

Mathematischer Bücherschatz. Systematisches Verzeichnis der wichtigsten deutschen und ausländischen Lehrbücher und Monographien des 19. Jahrhunderts auf dem Gebiete der mathematischen Wissenschaften. Von E. WÖLFFING. I Teil. *Reine Mathematik.* Leipzig, Teubner, 1903. Pp. xxxii + 416. Price 14 Marks.

ONE of the most valuable contributions that can be made in any science is a good bibliography, and the difficulty of making it is commensurate with its value. The preparation of such a work is an attractive field for a bibliophile, and many have been the attempts to enter it, but only a few who have made the venture have accomplished anything of permanent value. How difficult even the masters have found the task is easily seen by any scholar who has studied carefully some portion of the field, for his own bibliography soon becomes far more complete than any other that he consults on that particular domain.

For this reason it is easy to find fault with a work like the one under review. The task is so herculean, the patience needed is so inexhaustible, the scholarship demanded is so extensive, that errors are certain to creep in; errors of the pen, of judgment, of typography, and of knowledge. And recognizing all this, a reviewer should approach his examination of such a work with abundant charity, and with the determination to find the good that is in the book.

Approaching this work in such a spirit, Professor Wölffing is to be thanked for having compiled such an extensive list of books, for having attempted to classify them, and for having given two indexes, one by subjects and one by authors.

A rough estimate shows that there must be about twenty thousand works catalogued, and to have that number of titles arranged in fairly convenient form is helpful. Even more valuable, in many ways, is the preface, in which is given a rather extended list of sources for the construction of a bibliography. Scholars who fail to find what they wish in this work, thus have other sources suggested, which in turn may lead to the information desired.

Not much else can be said for the book. It abounds in errors of every kind, and the author's list of over two hundred

"Verbesserungen" could easily be increased to proportions that would be alarming.

In the first place the classification is not altogether satisfactory, arranged as it is in over three hundred sub-heads. But the arbitrary way in which books have been placed under these heads is inexplicable. For example, under Algorithm are about a dozen works, evidently selected for the sole reason that they chance to have this word in their titles. The list includes an historical sketch of Grammateus, a work on the calculus, a treatise on higher numerical equations, a text-book on arithmetic, and a work on continued fractions. The list of text books is equally ill-considered, some of our most mediocre English and American works appearing, often forgotten lumber of the past, while algebras like those of Newcomb, and Oliver, Wait and Jones, are overlooked. Even Robinson's old algebra, and Olney's works, good text-books in their day, are not mentioned, although Perkins's algebra (Utica, 1842) has a place. The same lack of any well-defined basis of selection is apparent in almost any list that one selects. In the pedagogy of mathematics, for example, if such elementary works as those of Kallas, Harms, and Grass are to have place, why should not Grube, Diesterweg, and Knilling be mentioned? If DeMorgan's work on the study and difficulties of mathematics is to be included, why not Lagrange's lectures on elementary mathematics?

Among the vagaries of classification, Dixon's Foundations of geometry appears under philosophy of mathematics, as published at Cambridge at 6 shillings, although Hilbert's work with the same title is not given in this group. Dixon again appears under principles of geometry, along with Hilbert, but this time as published at Cambridge at 7.2 marks. Comte's Philosophy of mathematics, as translated by Gillespie, appears on page 6 (N. Y., 1851, $1.50), and Gillespie's Philosophy of mathematics (the same American edition) on the following page as published in Leipzig (1852) at 8 shillings. The fact that confusion as to place of publication should occur in the case of New York, London, and Oxford or Cambridge, is natural, as in the case of Dodgson's Pillow problems (referred to New York), but to see English books frequently quoted in dollars, and American books in shillings, strikes an English reader as peculiar.

The incompleteness of the lists is evident on the most cursory

inspection. An examination of the bibliography in Ahrens's Mathematische Unterhaltungen und Spiele, to take a relatively unimportant case, shows how weak is § 313, Mathematische Belustigungen.

The misprints are too numerous to consider, except as types. T. L. Heath appears as R. D. Heath, author of a life of Appollonius, with no mention of his other works. McCormack's translation of Schubert, published at 75 cents, is assigned to MacCormack and the price is given as $3.75. Professor John Dewey appears as A. Dewey, Professor Cajori as Cajory, a good English Euclid as Euclide, H. N. Robinson as H. or N. H. Robinson, and G. A. Wentworth as G. H. Wentworth. Arithmetic appears as arithmetics, "another" as "an other" (page 192), and McLellan as MacLellan (page 8), and the many other errors of this kind in English show that similar ones may be expected in other languages. The same carelessness is seen in the index and the cross references, there being no mention of Lagrange (as indexed) on page 139, the Braunmühl cross reference (page 1) 188 being an error for 181, and the reference to page 357 in the Inhaltsverzeichnis being an error for 358.

Since a definite basis of classification, a careful selection of material, and a minimum of typographical errors are essential if a work of this kind is to be recognized, there can be but one opinion of this attempt of Professor Wölffing, that it will take no rank as an authority in the field of mathematical bibliography.

DAVID EUGENE SMITH.

Elemente der Vektor-Analysis. Von A. H. BUCHERER. Leipzig, B. G. Teubner, 1903. vi + 91 pp.

VECTOR analysis has been attracting to itself each year more and more attention in Germany. Up to the present year no account of the subject had been published in separate form except the bulky classics little suited to the actual needs of the practical physicist. Dr. Bucherer has supplied the deficiency admirably so far as he goes. It is only natural to compare the book with the very similar introductory chapter of Professor Föppl's Einführung in die Maxwell'sche Theorie der Elektricität (1894). In fact the two presentations of vector analysis cover the same number of pages and where Bucherer surpasses in quantity Föppl excels in clearness.

The subjects treated are as usual: scalar and vector products; differentiation with respect to a scalar d/dt and with respect to

space by the operator "del," ∇; the properties of the divergence and curl and the combination of them with the line, surface or space integrals to yield the theorems of Gauss, Stokes, and Green. The potential, the equations of Laplace and Poisson, the theorems of Beltrami, of Poincaré and Lorentz, and the principle of Huygens stated in mathematical form close the volume. The applications to physics consist of only the merest fragments of mechanics and hydromechanics. It might have been better to enlarge the work a trifle so as to find room for some slight mention of electricity and magnetism — such as the determination of the Heaviside-Hertz equations of the electromagnetic field. To have taken up the question of the propagation of light in crystalline bodies would have been impossible owing to the complete omission of the elements of the theory of the linear vector function. It may be doubted whether this omission, which deprives the student of perhaps the greatest advantages of the vector notation, is wise.

The treatment of the author may be characterized as formal rather than strictly rigorous—the "physicist's" way of reasoning is employed throughout. For instance, the resolution of a vector into its solenoidal and irrotational parts is worked out without particularly mentioning the fact that such resolution is impossible unless the vector vanishes everywhere at infinity. Such things are, however, not likely to be heeded by those for whom the book is intended.

In selecting a notation the author decides in favor of Heaviside's as modified by Föppl. German letters are used for vectors, Latin for scalars. The scalar product is denoted by juxtaposition of the letters, the vector product by prefixing a a V of unusual font. This question of notation is serious. The most recent German opinion on the subject is by Prandtl in the *Jahresbericht der Mathematiker-Vereinigung* for September. He argues for the adoption of the dot and cross of J. Willard Gibbs with the use of heavy faced type for vectors. The usual objection that in work at the board it is difficult to distinguish between light and heavy type may be met satisfactorily by the use of script for vectors and printed letters for scalars.

On the next page is a table of the notations which the reviewer has seen recently in use or suggested by advocates of vector notations. As a word of explanation it may be stated that only the analytic symbols have been set down. Many persons use the

	Cartesian Equivalent.	Grassmann.	Hamilton and Followers.	Gibbs.	Heaviside.	Föppl, Bucherer.	Henrici.
Scalars.	$a, b, c, \cdots$	Italic.	Italic.	Italic.	Italic.	Italic.	Italic.
Vectors.	a_x, a_y, a_z,	Italic.	Greek.	Clarendon.	Clarendon.	German.	Greek.
Scalar Product.	$a_xb_x + a_yb_y + a_zb_z$	$[a \mid b]$	$-Sa\beta$	$\mathbf{A}\cdot\mathbf{B}$	$\mathbf{AB}$	$\mathfrak{AB}$	$(a\beta)$
Vector Product.	$a_yb_z - a_zb_y, \cdots$	$[ab]$	$Va\beta$	$\mathbf{A}\times\mathbf{B}$	$V\mathbf{AB}$	$V\mathfrak{AB}$	$[a\beta]$
Quaternion Product.			$a\beta$				
Dyad Product.	$a_xb_x, a_xb_y, \cdots$			$\mathbf{AB}$	$\mathbf{A}\cdot\mathbf{B}$		
Gradient of V.	$\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}$		∇V	∇V	∇V	∇V	∇V
Divergence of Vector Function $\mathbf{V}$, or $\mathfrak{V}$, or V_x, V_y, V_z or η.	$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$		$-S\nabla\eta$	$\nabla\cdot\mathbf{V}$		$\nabla\mathfrak{V}$	$(\nabla\eta)$
Curl.	$\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, \cdots$		$V\nabla\eta$	$\nabla\times\mathbf{V}$		$V\nabla\mathfrak{V}$	$[\nabla\eta]$
Linear Vector Function Operator.	Determinant.		φa Operator inseparable from Operand.	ϕ	φ or c		

terms "curl" and "div," which are scarcely to be ranked as symbols. It may be seen that the system of the late Professor Gibbs is perhaps the only one in which the matter of a complete, consistent, and practical notation has been carefully thought out to the end.

E. B. WILSON.

Wissenschaftliche Grundlagen der Elektrotechnik. Von GALILEO FERRARIS. Deutsch herausgegeben von LEO FINZI. Leipzig, B. G. Teubner, 1901. xii + 358 pp.

THE title of this work could not have been better chosen. The lectures on which the book is founded were delivered at the Reale Museo Industrale at Turin and were intended for technical students. Yet the scientific foundations of electric, magnetic, and electromagnetic theory have been placed so emphatically in the foreground and the details of the applications so thoroughly omitted that the work serves also the purpose of an introduction to modern electrical theories. The style is everywhere of the simplest with the emphasis always on the physical conceptions involved. The author begins without assuming on the part of the reader any greater preparation than the elements of the calculus and he builds up carefully, one at a time, the conceptions connected with vectors, electricity, magnetism, and electromagnetism, until at the close of the book he is able to present the elements of Maxwell's theory.

The chief original work of Ferraris was on the magnetic fields arising from the composition of alternating currents of different phases. His memoir entitled "Rotazioni elettrodinamiche," which was published in 1888, is a classic and of value no less for its practical applications to the construction of alternating current motors than for its theory. One might look for an elaborate treatment of this subject in the book before us. As a matter of fact only small mention of it is made, doubtless because the author has in mind the necessity of laying the general foundations rather than of going into details even upon the questions which might interest him most.

The concluding chapter on Maxwell's theory, Hertz's experiments, and Poynting's theorem could have been rendered even more clear by a greater insistence on the notions of curl and divergence; but as it stands it is useful as a guide for those who would follow the theory in its original Cartesian form. The appendix in which are discussed the scientific and practical

C.G.S. systems of units and the dimensions of the quantities occurring in electromagnetism is worthy of the rest of the book and a fitting end to a work which has seemingly no fault and a multitude of virtues.

E. B. WILSON.

NOTES.

AT the fourth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY the following officers were elected for the year 1904: chairman, Professor R. E. ALLARDICE; secretary, Professor G. A. MILLER; members of the programme committee, Professors M. W. HASKELL, IRVING STRINGHAM, and G. A. MILLER.

THE opening (January) number of volume 5 of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "The subgroups of order a power of 2 of the simple quinary orthogonal group in the Galois field of order $p^n = 8l \pm 3$," by L. E. DICKSON; "On certain invariants of two triangles," by J. G. HUN; "Isothermal systems of geodesics," by E. KASNER; "Zur Gruppentheorie, mit Anwendungen auf die Theorie der linearen homogenen Differentialgleichungen," by A. LOEWY; "On the group of the sign $(0, 3; 2, 4, \infty)$ and the functions belonging to it," by J. W. YOUNG; "On the definition of reducible hypercomplex number systems," by S. EPSTEEN; "A simple proof of a theorem in the calculus of variations (extract from a letter to Mr. W. F. Osgood)," by E. GOURSAT.

WITH the January issue the *Mathematical Supplement of School Science* assumes an independent existence as a bimonthly journal under the title *School Mathematics*. It remains under the responsible editorship of Professor G. W. MYERS, who will be assisted by a board of associate editors. It is intended to make the journal an exponent of the teaching of mathematics in the United States.

BEGINNING with the January number, there will appear in the *American Mathematical Monthly* a series of articles by Dr. E. B. WILSON on "Spherical geometry," presented in the spirit

of the recent investigations on the foundations of geometry and, in particular, as an introduction to non-euclidean geometries. The series is based on the writer's course to seniors at Yale University for the purpose of training students to become effective teachers of geometry in the secondary schools.

PROFESSOR EMIL LAMPE, of Charlottenberg, for over twenty years editor of the *Jahrbuch über die Fortschritte der Mathematik*, feels compelled to retire from the editorship and has requested the Deutscher Mathematiker-Vereinigung to make provision to have the publication of the *Jahrbuch* continued.

AT the meeting of the American association for the advancement of science, held at St. Louis, December 26, 1903–January 1, 1904, Professor ALEXANDER ZIWET was elected vice-president of Section A (mathematics and astronomy). Professor L. G. WELD retains the office of secretary of Section A. The association will hold its next annual meeting at Philadelphia, under the presidency of Professor W. G. FARLOW. A report of the St. Louis meeting of Section A will appear in the March number of the BULLETIN.

AT the meeting of the London mathematical society held on December 10, the following papers were read: By Mr. R. J. DALLAS, "Proof of a formula in elliptic functions"; by Professor A. C. DIXON, "On many-valued Newtonian potentials"; by Rev. F. H. JACKSON, "A generalization of Neumann's expansion of an arbitrary function in a series of Bessel's functions"; by Dr. E. W. HOBSON, "Modes of convergence of infinite series of functions of a real variable"; by Professor J. D. EVERETT, "On normal and antinormal piling"; by Mr. W. H. YOUNG, "On the distribution of the points of uniform convergence of a series of functions"; by Mr. E. B. JOURDAIN, "On functions all of whose singularities are non-essential."

IN connection with the third international congress of mathematicians, to be held at Heidelberg, August 8–13, 1904, it is planned to have an exhibition of the more important mathematical literature of the last ten years. Societies and individual mathematicians have been invited to contribute to the exhibition. Professor A. GUTZMER, of the University at Jena, is chairman of the committee in charge.

At the general sessions of the congress, besides the address by Professor KOENIGSBERGER in commemoration of Jacobi, four addresses, in German, English, French, and Italian, will be

made by Professors WIRTINGER, GREENHILL, DARBOUX, and SEGRE.

In response to the invitations issued by the committee of arrangements, nearly 500 persons have already signified their intention to attend the congress.

AT the Cassel meeting of the Deutscher Mathematiker-Vereinigung a committee on mathematical bibliography was appointed, who will soon publish a list of rare mathematical books and periodicals, and of the various libraries in which they are to be found. In order to make this list as useful and complete as possible, the committee requests those who have information of rare important mathematical works to communicate with any of its members: Professor FELIX MÜLLER, Friedenau-Berlin, Rönnebergstrasse 16; Professor E. WÖLFFING, Stuttgart, Hackländerstrasse 38; Professor A. GUTZMER, Jena, Schäfferstrasse 4.

THE universities named below offer the following courses for the summer semester of 1904:

UNIVERSITY OF GÖTTINGEN. — By Professor F. KLEIN: Differential equations, four hours; Seminar, two hours.—By Professor D. HILBERT: Theory of functions, four hours; Exercises in theory of functions, two hours; Conception of number and quadrature of the circle, two hours. By Professor H. MINKOWSKI: Geometry of the line and sphere, four hours; Mechanics of continua, two hours; Exercises in theory of functions (with Professor Hilbert), two hours. — By Professor M. BRENDEL: Mathematics of insurance, four hours; Theory of probabilities, two hours; Exercises in insurance and probabilities, two hours.—By Professor F. SCHILLING: Analytic geometry, four hours; Exercises in descriptive geometry, and perspective, two hours. — By Dr. E. ZERMELO: Curves and surfaces, four hours. — By Dr. O. BLUMENTHAL: Differential and integral calculus I, four hours; Exercises, two hours.

UNIVERSITY OF INNSBRUCK. — By Professor OTTO STOLZ: Differential and integral calculus of real functions (continuation), with exercises, four hours; Theory of functions of a complex variable according to Cauchy and Weierstrass, with exercises, three hours. — By Professor K. ZINDLER: Elements of the theory of numbers, three hours; Applications of the calculus to geometry, with exercises, four hours.

UNIVERSITY OF KÖNIGSBERG. — By Professor F. MEYER: Theory of numbers, four hours. — By Professor A. SCHOENFLIES: Analytic geometry, four hours. — By Professor L. SAALSCHÜTZ: Differential calculus, four hours. — By Dr. T. VAHLEN: Algebra, four hours.

UNIVERSITY OF WÜRZBURG. — By Professor F. PRYM: Integral calculus, six hours; Exercises, two hours. — By Professor E. SELLING: Partial differential equations, four hours; Analytic mechanics, four hours. — By Professor G. ROST: Applications of the calculus to the theory of plane curves, four hours; analytic and synthetic geometry of conics, four hours; Theory of twisted curves and surfaces, four hours; Theory of elliptic functions, two hours.

AT the annual public meeting of the Paris academy of sciences, December 21, 1903, the following mathematical prizes were awarded: Francoeur prize, 1,000 francs, ÉMILE LEMOINE; Poncelet prize, 2,000 francs, DAVID HILBERT; d'Ormoy prize, 10,000 francs, JACQUES HADAMARD; Saintour prize, 3,000 francs, MARCEL BRILLOUIN; Extraordinary prize of 6,000 francs, divided between MM. MAUGAS, JEHENNE, GAILLARD, and GERMAIN.

DR. C. STÖRMER, of the University of Christiania, has been promoted to a professorship of mathematics.

PROFESSOR S. A. F. WHITE, of Wadham College, Oxford, has been appointed to the professorship of mathematics, and Professor J. B. DALE, of St. John's, Cambridge, to an assistant professorship of mathematics, in King's College, London.

PROFESSOR L. BOLTZMANN of Vienna has been elected honorary member of the Academy of science at Moscow.

PROFESSORS D. HILBERT, of Göttingen, and H. WEBER, of Strassburg, have been elected corresponding members of the Academy of science at Munich.

PROFESSOR F. W. HANAWALT, of Iowa Wesleyan University, has been appointed to the professorship of mathematics and astronomy at Albion College, Michigan.

PROFESSOR R. J. ALEY, of the University of Indiana, has been elected to the editorship of the *Educator-Journal.*

DR. SAUL EPSTEEN, of the University of Chicago, has been made an editor of the *American Mathematical Monthly.*

DR. G. A. BLISS, instructor in mathematics at the University of Minnesota, has been appointed associate in mathematics in the University of Chicago.

MR. W. J. RISBY has been appointed instructor in mathematics at the Armour Institute, Chicago.

REV. GEORGE SALMON, provost of Trinity College, Dublin, died January 23.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ANDRAE (A.). Hilfsmittel zu einer allgemeinen Theorie der linearen elliptischen Differentialgleichung zweiter Ordnung. (Diss.) Göttingen, 1903. 8vo. 111 pp.

BJERKNES (V.). Till minde om Carl Anton Bjerknes. Christiania, 1903. 8vo. 24 pp.

BOCCARDINI (G.). See SACCHERI (G.).

CAMPBELL (J. E.). Introductory treatise on Lie's theory of finite continuous transformation groups. New York, Oxford University Press, 1903. 8vo. 20 + 416 pp. Cloth. $4.75

CASTELNUOVO (G.). Lezioni di geometria analitica e proiettiva. Vol. 1. Milano, 1903. 8vo. 516 pp. Fr. 12.00

CHEMIN (O.). See SALMON (G.).

COMBEROUSSE (C. DE). Algèbre supérieure, à l'usage des candidats à l'Ecole polytechnique, à l'Ecole normale supérieure, à l'Ecole centrale. Ie partie: Compléments d'algèbre élémentaire (déterminants, fractions continues, etc.); combinaisons; séries; étude des fonctions; dérivées et différentielles; premiers principes du calcul intégral. 3e édition. Paris, Gauthier-Villars, 1904. 8vo. 21 + 767 pp. (Tome III du Cours de mathématiques.) Fr. 15.00

DASSEN (C. C.). Etude sur les quantités mathématiques. Grandeurs dirigées. Quaternions. Paris, Hermann, 1903. 8vo. 6 + 135 pp.

ERLER (W.). Die Elemente der Kegelschnitte in synthetischer Behandlung, zum Gebrauche in der Prima höherer Lehranstalten bearbeitet. 6te Auflage, besorgt von L. Huebner. Leipzig, Teubner, 1903. 8vo. 6 + 60 pp. Boards. M. 1.20

GAUSS (C. F.). Werke. Vol. IX. Herausgegeben von der königl. Gesellschaft der Wissenschaften zu Göttingen. Leipzig, 1903. 4to. 528 pp. Boards. M. 26.00

GEIGENMÜLLER (R.). Differential and integral calculus; a brief and simple introduction to higher mathematics, for self-instruction. Translated by M. Likharev. St. Petersburg, 1904. 4to. 95 pp. (Russian.) M. 4.50

GRÜNBAUM (H.). Isolierte und reine Gruppen und die Marbe'sche Zahl p. Kritische Studie zur Wahrscheinlichkeitslehre. Würzburg, 1903. 8vo. M. 1.20

GÜNTHER (N. M.). On the theory of algebraic forms in connection with the integration of linear differential equations. St. Petersburg, 1903. 8vo. 234 pp. (Russian.)

HEGER (R.), HENKE (R.). See SCHLOEMILCH.

HENSEL (K.). See KRONECKER (L.).

HILBERT (D.). Grundlagen der Geometrie. 2te, durch Zusätze vermehrte und mit fünf Anhängen versehene Auflage. Leipzig, Teubner, 1903. 8vo. 5 + 175 pp. M. 5.20

HOCHHEIM (A.). Aufgaben aus der analytischen Geometrie der Ebene. Heft I: Die gerade Linie, der Punkt, der Kreis. B. Auflösungen. 3te, vermehrte Auflage. Leipzig, Teubner, 1904. 8vo. 8 + 128 pp. Cloth. M. 2.60

HUEBNER (L.). See ERLER (W.).

HUNTINGTON (E. V.). Complete sets of postulates for the theory of real quantities. 4to. (*Transactions of the American Mathematical Society*, Vol. 4, pp. 358–370.)

——. Two definitions of an abelian group by sets of independent postulates. Definitions of a field by sets of independent postulates. (*Transactions of the American Mathematical Society*, Vol. 4, pp. 27–37.)

——. See POINCARÉ (H.).

JUNKER (F.). Höhere Analysis. Teil 2: Integralrechnung. 2te, verbesserte Auflage. 2ter Abdruck. Leipzig, Göschen, 1903. 12mo. 208 pp. Cloth. (Sammlung Göschen, No. 88.) M. 0.80

KLÖRES (C.). Zur Geschichte der Steiner'schen Konstruktion einer Fläche 2. Ordnung. (Diss.) Rostock, 1903. 8vo. 39 pp. 2 plates.

KOSSOW (F.). Zur Scheitelpunktsbestimmung des Paraboloids. (Diss.) Rostock, 1903. 8vo. 58 pp.

KRAFT (A.). Ueber ganze transzendente Funktionen von unendlicher Ordnung. (Diss.) Göttingen, 1903. 8vo. 76 pp.

KRONECKER (L.). Vorlesungen über Mathematik. (In 2 Teilen.) Teil II: Vorlesungen über allgemeine Arithmetik. Abschnitt 2: Vorlesungen über die Theorie der Determinanten, bearbeitet und fortgeführt von K. Hensel. Band 1: 1ste bis 21ste Vorlesung. Leipzig, Teubner, 1903. 8vo. 12 + 390 pp. M. 20.00

LIKHAREV (M.). See GEIGENMÜLLER (R.).

MEYER (R.). See ZEUTHEN (H. G.).

NEUBERG (J.). Cours d'algèbre supérieure. Liège, 1902. 8vo. 279 pp. Fr. 5.00

NICCOLETTI (O.). Lezioni di algebra complementare. Pisa, 1903. 8vo. 624 pp. (Lith.) Fr. 10.00

OBLOMIEVSKY (D. D.). Symmetric functions. Edited by D. F. Selivanov. St. Petersburg, 1903. 8vo. 163 pp. (Russian.)

OCAGNE (M. D'). Exposé synthétique des principes fondamentaux de la nomographie. Paris, Gauthier-Villars, 1903. 4to. 64 pp. Fr. 3.50

POINCARÉ (H.). Review of Hilbert's "Foundations of geometry." Translated by E. V. Huntington. (*Bulletin of the American Mathematical Society*, Vol. 10, pp. 34–49.) New York, 1903. 8vo. 23 pp.

PUCKLE (G. H.). Elementary treatise on conic sections and algebraic geometry: examples and hints for their solution. For beginners. New edition. London, Macmillan, 1903. 8vo. 388 pp. Cloth. 7s. 6d.

RABINOVITCH (I. E.). The foundations of the Euclidian geometry as viewed from the standpoint of kinematics. (Diss., Johns Hopkins University, April 29, 1901.) New York, published by the author, 1903. 8vo. 12 + 116 pp.

REINECKE (W.). Die Grundlagen der Geometrie nach Kant und neueren Autoren. Teil I. (Diss.) Halle a. S., 1903. 8vo. 57 pp.

RICHARD (J.). Sur la philosophie des mathématiques. Paris, Gauthier-Villars, 1903. 8vo. 248 pp. Fr. 3.25

ROTHE. Lösung einiger Aufgaben über Flächenberechnungen mit Hilfe elliptischer Integrale. Teil II. Nordhausen, 1903. 4to. 33 pp. M. 2.00

SACCHERI (G.). L'Euclide emendato. Traduzione e note di G. Boccardini. Milano, Hoepli, 1904. 16mo. 24 + 126 pp. (Manuali Hoepli.)

SALMON (G.). Traité de géométrie analytique à trois dimensions, traduit de l'anglais sur la 4e édition par O. Chemin. (En 3 volumes.) Vol. II: Théorie des surfaces; courbes gauches et surfaces développables; familles de surfaces. 2e édition. Paris, Gauthier-Villars, 1903. 8vo. Fr. 6.00

SCHLOEMILCH'S Handbuch der Mathematik. 2te Auflage, herausgegeben von R. Henke und R. Heger. Vol. I: Elementarmathematik. 12 + 611 pp. Vol. II: Höhere Mathematik. 1ster Teil. 8 + 765 pp., 12 plates. Leipzig, Barth, 1904. 8vo. M. 40.00

SCHMID (E.). Die Cauchy'sche Methode der Auswertung bestimmter Integrale zwischen reellen Grenzen. (Diss.) Marburg, 1903. 8vo. 93 pp.

SELIVANOV (D. F.). See OBLOMIEVSKY (D. D.).

STEPHANSEN (M. A. E.). Ueber partielle Differentialgleichungen vierter Ordnung, die ein intermediäres Integral besitzen. (Diss., Zürich, begutachtet von Prof. Burkhardt.) Kristiania, Cammermeyer, 1902. 8vo. 80 pp.

ZEUTHEN (H. G.). Forelaesninger over mathematikens historie. II. 16. og 17. aarhundrede. Kjöbnhavn, 1903. 8vo. 624 pp. Kr. 12.00

——. Geschichte der Mathematik im XVI. und XVII. Jahrhundert. Deutsche Ausgabe, unter Mitwirkung des Verfassers bearbeitet von R. Meyer. Leipzig, Teubner, 1903. 8vo. 8 + 434 pp. (Abhandlungen zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Heft 17.) M. 16.00

II. ELEMENTARY MATHEMATICS.

ALEXANDROFF (I.). Aufgaben aus der niederen Geometrie, nach Lösungsmethoden geordnet und zu einem Uebungsbuche zusammengestellt. Mit einem Vorwort von M. Schuster. Leipzig, Teubner, 1903. 8vo. 6 + 123 pp M. 2.40

AMALDI (U.). See ENRIQUES (F.).

ARNOUX (V.). 2500 problèmes d'arithmétique et de géométrie pratique. Problèmes d'examen. Livre de l'élève. 9e édition, revue et augmentée. Paris, Larousse, 1903. 12mo. 288 pp. Fr. 2.00

BAKER (A.). Elementary plane geometry. Boston, Ginn, 1903. 146 pp. $0.50

BARDEY (E.). Arithmetische Aufgaben nebst Lehrbuch der Arithmetik, vorzugsweise für höhere Bürgerschulen, Realschulen, Progymnasien und Realprogymnasien. 13te Auflage, in der neuen Rechtschreibung. Leipzig, Teubner, 1903. 8vo. 9 + 269 pp. Cloth. M. 2.40

BARRELL (F. R.). Elementary geometry. Sections 1 and 2. London, Longmans, 1903. 8vo. Cloth. 3s. 6d.

BORK (H.), CRANTZ (P.) und HAENTZSCHEL (E.). Mathematischer Leitfaden für Realschulen. Teil 2: Trigonometrie und Stereometrie. 3te, verbesserte Auflage. Leipzig, Dürr, 1904. 8vo. 128 pp. Cloth. M. 1.60

BOS (P. J.). Leerboek der algebra. Theorie en opgaven. 5te, vermeerderde druk. Deel II. Harlem, 1903. 8vo. 148 pp. M. 2.50

BOURDON. Trigonométrie rectiligne et sphérique. Edition revue et annotée par C. Brisse. Nouveau tirage. Paris, Gauthier-Villars, 1904. 8vo. 131 pp. Fr. 3.00

BOURLET (C.). Eléments d'algèbre, contenant 631 exercices et problèmes, rédigés conformément aux programmes du 31 mai 1902 (premier et deuxième cycles). Paris, Hachette, 1903. 16mo. 314 pp. Fr. 2.00

BRISSE (C.). See BOURDON.

BRUTON (F. A.). Victoria preliminary papers in mathematics and mechanics. Answers to all examples, solutions to geometry papers, etc. London, 1903. 8vo. 150 pp. 1s. 6d.

BURKITT (F. G.). Tables of logarithms and decimals adapted to business and statistical calculations. London, 1903. 8vo. Cloth. 1s.

CASTLE (F.). Manual of practical mathematics. London and New York, Macmillan, 1903. 12mo. 11 + 541 pp. Cloth. $1.50

COPPO (A.). Nozioni di aritmetica e di geometria per la quarta e quinta classe elementare, secondo i programmi 29 novembre 1894. Torino, 1904. 8vo. 64 pp. Fr. 0.60

CRANZ (P.). See BORK (H.).

DEAKIN (R.). Euclid, books 1, 2. With an introductory course of drawing and measurement, and problems on practical geometry. 3d imprint, 2d edition. London, Clive, 1903. 8vo. 188 pp. Cloth. 1s.

DIESENER (H.). Die Buchstabenrechnung und Algebra, einschliesslich der Logarithmen und des Rechnens mit denselben; mit einer grossen Zahl vollständig ausgerechneter Beispiele und Aufgaben, bearbeitet für den Selbstunterricht und zum Gebrauche an Bau-, Gewerbe- und Fortbildungsschulen. 3te Auflage. Halle, Hofstetter, 1903. 8vo. 4 + 276 pp. M. 4.00

ENRIQUES (F.) e AMALDI (U.). Elementi di geometria ad uso delle scuole normali. Bologna, Zanichelli, 1903. 16mo. 7 + 271 pp. Fr. 2.50

EUCLID. See DEAKIN (R.).

EUKLIDES' fyra första böcker. Med smärre förändrigar och tillägg utgifven af C. F. Lindman. 11te upplaga. Stockholm, 1903. 8vo. 125 pp. Cloth. M. 2.50

GAGOLEV (A. N.). Collection of geometrical problems. 2d edition. Moscow, 1903. 8vo. 288 pp. (Russian.) M. 2.00

GAMBIOLI (D.). See RIBONI (G.).

GHERARDI (U.). Nozioni d'aritmetica e di geometria per gli alunni della prima classe ginnasiale, in conformità ai programmi ministeriali vigenti. 3a edizione riveduta. Vol. I. Milano, Trevisini, 1903. 16mo. 151 pp. Fr. 1.50

GRÉVY (A.). Trigonométrie, à l'usage des élèves des classes de seconde et de première C et D, et de mathématiques A et B (programmes du 31 mai 1902). Paris, Nony, 1903. 16mo. 8 + 272 pp. Fr. 2.25

GUICHARD (C.). Traité de géométrie. Première partie: géométrie élémentaire (plane et dans l'espace). 2e édition. Paris, Nony, 1903. 2 + 457 pp. Fr. 5.00

HAENTZSCHEL (E.). See BORK (H.).

HALL (H. S.) and STEVENS (F. H.). A school geometry. Part 4, containing the substance of Euclid, book 2, with book 3, 35–37. New York, Macmillan, 1903. 12mo. $0.20

HUEBNER (E.). Auswahl mathematischer Aufgaben für Prima. Teil II. Königsberg, 1903. 8vo. 23 pp. M. 1.00

KISELEV (A.). Elementary Geometry. With appendix on the most important methods for solving geometrical problems. 11th edition. Moscow, 1903. 8vo. 307 pp. (Russian.) M. 3.50

LAMÉ (G.). Examen des différentes méthodes employées pour résoudre les problèmes de géométrie. Paris, Courcier, 1818. 8vo. 12 + 124 pp. Ré-impression facsimilé. Paris, Hermann, 1903. Fr. 5.00

LINDMAN (C. F.). See EUKLIDES.

MALVASI (G.). Equazioni a coefficienti proporzionali. Bologna, Cuppini, 1903. 8vo. 23 pp.

MARAKUIEV (N. N.). Elementary algebra. Part 1: Theory. 2d edition, revised. Moscow, 1903. 8vo. 958 pp. (Russian.) M. 16.00

MARKS (C. I.). Mathematical questions and solutions. New series. Vol. 4. London, Hodgson, 1903. 8vo. Cloth. 6s. 6d.

MARTINI ZUCCAGNI (A.). Trattato di geometria elementare ad uso dei licei. Livorno, Giusti, 1903. 8vo. 10 + 334 pp. Fr. 2.60

——. Trattato di trigonometria piana e sferica ad uso degli istituti tecnici. Livorno, Giusti, 1903. 16mo. 11 + 228 pp. Fr. 2.50

NELSON (T.). Concrete geometry; introductory to formal plane geometry. Concord, Rumford, 1903. 12mo. 8 + 140 pp. Cloth. $0.60

NIEWENGLOWSKI. Tables de logarithmes à cinq décimales. 2e édition. Paris, 1903. 8vo. Cloth. Fr. 3.00

RIBONI (G.). Elementi di geometria a uso delle scuole secondarie inferiori, corredati da una raccolta di circa mille esercizi per cura di D. Gambioli. Bologna, Zanichelli, 1903. 16mo. 8 + 505 pp. Fr. 2.40

SCHMID (A.). See SICKENBERGER (A.).

SCHUBERT (H.). Vierstellige Tafeln und Gegentafeln für logarithmisches und trigonometrisches Rechnen, in zwei Farben zusammengestellt. 2te Auflage. Leipzig, Göschen, 1903. 12mo. 128 pp. Cloth. (Sammlung Göschen, No. 81.) M. 0.80

SCHUSTER (M.). See ALEXANDROFF (I.).

SICKENBERGER (A.). Leitfaden der Arithmetik, nebst Uebungsbeispielen. 9te, vermehrte Auflage, bearbeitet von A. Schmid. München, Ackermann, 1904. 8vo. 6 + 196 pp., 1 plate. M. 1.60

SIGÜENZA Y GARRIDO (D. J.). Tratado de trigonometría. Madrid, Cuerpo de Artillería, 1903. 8vo. 8 + 278 pp. Fr. 6.00

STEVENS (F. H.). See HALL (H. S.).

VINTÉJOUX (F.). Eléments d'arithmétique, de géométrie et d'algèbre. 6e édition, revue. Paris, Hachette, 1903. 16mo. 7 + 575 pp. Fr. 2.50

WACKERBARTH (A. F. D.). Femstelliga logarithmatabeller. 3te upplaga. Stockholm, 1903. 18 + 228 pp. Cloth. M. 3.50

WENTWORTH (G. A.). New plane and spherical trigonometry, surveying and navigation. Teachers' edition. Boston, Ginn, 1903. 12mo. 2 + 390 pp. Cloth. $1.20

WIENECKE (E.). Der geometrische Vorkursus in schulgemässer Darstellung. Mit reichem Aufgabenmaterial nebst Resultaten, zum Gebrauch an allen Lehranstalten. Leipzig, Teubner, 1904. 8vo. 4 + 97 pp. Boards. M. 2.20

WROBEL (E.). Uebungsbuch zur Arithmetik und Algebra, enthaltend die Formeln, Lehrsätze und Auflösungsmethoden in systematischer Anordnung und eine grosse Anzahl von Fragen und Aufgaben. Anhang für höhere realistische Lehranstalten (Realgymnasien, Oberrealschulen u. s. w.) 3te, durchgesehene Auflage, in neuer Rechtschreibung. Rostock, Koch, 1904. 8vo. 3 + 72 pp. M. 0.80

III. APPLIED MATHEMATICS.

APPELL (P.). Traité de mécanique rationnelle. 2e édition, entièrement refondue. Vol. II: Dynamique des systèmes; mécanique analytique. Paris, Gauthier-Villars, 1904. 8vo. 8 + 552 pp. Fr. 16.00

BRIGGS (W.) and BRYAN (G. H.). Tutorial statics. 3d edition, enlarged. London, 1903. 8vo. 374 pp. Cloth. 3s. 6d.

BRYAN (G. H.). See BRIGGS (W.).

CHIPART (H.). La théorie gyrostatique de la lumière. Paris, Gauthier-Villars, 1904. 8vo. 196 pp. Fr. 6.50

CONSTAN (P.). Cours élémentaire d'astronomie et de navigation, à l'usage des capitaines au long cours et des élèves des écoles d'hydrographie. Vol. I: Astronomie. Paris, Gauthier-Villars, 1903. 8vo. 4 + 255 pp. Fr. 7.50

EICHHORN (A.). Der akustische Maasstab für die Projectbearbeitung grosser Innenräume, in seiner Beziehung zu den musikalischen Harmonien erläutert und nach seinen harmonischen Verhältnissen theoretisch berechnet und zeichnerisch dargestellt. 2te Auflage. Berlin, Spielmeyer, 1903. 8vo. 6 + 87 pp., 1 plate. M. 4.00

EWING (J. A.). Strength of materials. 2d edition. London, Clay, 1903. 8vo. 260 pp. Cloth. 12s.

GALLUSSER (H.) und HAUSMANN (M.). Theorie und Berechnung elektrischer Leitungen. Berlin, Springer, 1904. 8vo. 11 + 164 pp. Cloth. M. 5.00

GEISSLER (K.). Grundlagen der mathematischen Erdkunde. Leipzig, Teubner, 1903. 8vo. 6 + 199 pp.

GUICHARD (C.). Traité de mécanique. Première partie: cinématique, à l'usage des élèves des classes de première C et D. Paris, Nony, 1903. 8vo. 8 + 108 pp.

HAENTZSCHEL (E.). Das Erdsphäroid und seine Abbildung. Leipzig, Teubner, 1903. 8vo. 8 + 140 pp. M. 3.00

HAUBER (W.). Statik. Teil I: Die Grundlehren der Statik starrer Körper. Leipzig, Göschen, 1903. 12mo. 148 pp. Cloth. (Sammlung Göschen, No. 178.) M. 0.80

HAUSMANN (M.). See GALLUSSER (H.).

HILDEBRANDSSON (H.) et TEISSERENC DE BORT (L.). Les bases de la météorologie dynamique. Historique; état de nos connaissances. (En 2 volumes.) Vol. II. 6e livraison: Sur la circulation générale de l'atmosphère. Paris, Gauthier-Villars, 1903. 8vo. Fr. 2.00

HOFF (J. H. VAN'T). Vorlesungen über theoretische und physikalische Chemie. Heft 3: Beziehungen zwischen Eigenschaften und Zusammensetzung. 2te Auflage. Braunschweig, Vieweg, 1903. 8vo. 10 + 155 pp. M. 4.00

HOOPER (W. G.). Aether and gravitation. London, Chapman, 1903. 8vo. 376 pp. Cloth. 7s. 6d.

KECK (W.). Vorträge über Mechanik als Grundlage für das Bau- und Maschinenwesen. 3te, vermehrte und verbesserte Auflage, herausgegeben von L. Hotopp. (In 3 Bänden.) Vol. I: Mechanik starrer Körper. Hannover, Helwing, 1904. 8vo. M. 10.00

KLEIN (F.) und SOMMERFELD (A.). Ueber die Theorie des Kreisels. Heft III: Die störenden Einflüsse; astronomische und geophysikalische Anwendungen. Leipzig, Teubner, 1903. 8vo. Pp. 513–759, and 4 pp. M. 9.00

LAPLANCHE (G. C. DE). Unités électriques et unités mécaniques, et leurs relations. Traité élémentaire. Paris, Nony, 1903. 16mo. 158 pp.

LAUENSTEIN (R.). Die graphische Statik. Elementares Lehrbuch, für den Schul- und Selbstunterricht, sowie zum Gebrauch in der Praxis bearbeitet. 8te Auflage. Stuttgart, Bergsträsser, 1904. 8vo. 8 + 252 pp. M. 5.40

LEBON (E.). Géométrie descriptive et géométrie cotée, conformes aux programmes du 31 mai 1902 pour l'enseignement secondaire. (Classes de première C et D.) Paris, Delalain, 1903. 8vo. 126 pp. Fr. 2.50

LOEWY (A.). Versicherungsmathematik. Leipzig, Göschen, 1903. 12mo. 145 pp. Cloth. (Sammlung Göschen, No. 180.) M. 0.80

LÜBECK (O.). Mechanik I (Statik). Unterweisungen und Beispiele. 6te Auflage. Strelitz, Hittenkofer, 1903. 8vo. 79 pp. (Unterrichtswerke nach Methode Hittenkofer für den Selbstunterricht, Schul- und Bureaugebrauch, Lehrfach No. 60.) M. 5.00

MAURER (E. R.). Technical mechanics: a text-book for technical colleges. New York, Wiley, 1903. 8vo. 4 + 382 pp. Cloth. $4.00

NIETHAMMER (F.). Berechnung und Entwurf elektrischer Maschinen, Apparate und Anlagen, für Studierende und Ingenieure. (In 5 Bänden.) Vol. I: Berechnung und Konstruktion der Gleichstrom-Maschinen und Motoren. 1ste Hälfte: Elektrische Berechnung der Gleichstrom-Maschinen und Motoren. Stuttgart, Enke, 1903. 8vo. 284 pp. M. 8.00

OESTERLIN (H.). Untersuchungen über den Energieverlust des Wassers in Turbinenkanälen. Berlin, Springer, 1903. 8vo. 75 pp., 5 plates. M. 3.00

ORBAAN (C.). Praktisch handboekje voor het berekenen van verschillende premien en premiereserven van lebensverzekering. Amsterdam, 1903. 8vo. 9 + 99 pp. M. 2.50

OSTENFELD (A.). Teknisk statik. Grundlag for forelaesninger paa Polyteknisk Laereanstalt. Del. I. 1900. 500 pp. Del II. 1903. 408 pp., 24 plates. 8vo. M. 26.00

——. Technische Statik. Vorlesungen über die Theorie der Tragkonstruktionen. Deutsche Ausgabe, besorgt von D. Skouge. Leipzig, Teubner, 1904. 8vo. 8 + 457 pp., 33 plates. Cloth. M. 12.00

PATON (J.). Petit traité mathématique et pratique des opérations commerciales et financières. Vol. I: Arithmétique commerciale. Paris, 1903. 8vo. 406 pp.

PICART (L.). Sur quelques points de la théorie de la capture des comètes. Paris, Gauthier-Villars, 1903. 8vo. 24 pp. Fr. 2.00

PITTONI (L.). Principi di prospettiva. 2a edizione. Livorno, Giusti, 1904. 16mo. 6 + 71 pp. (Biblioteca degli studenti; riassunti per tutte le materie d'esame nei licei, ginnasi, istituti tecnici, ecc., Vol. 51.) Fr. 0.50

PRIESTER (H.). Distanzrelationen und Verwandtes. Langenberg, 1903. 8vo. 61 pp., 1 plate. M. 1.50

REPOSSI (F.). Elementi di prospettiva lineare, ad uso delle scuole tecniche e normali. 3a edizione nuovamente riveduta. Modena, Forghieri e Pellequi, 1903. 8vo. 63 pp.

ROMÁN (D. E.). Estudio del movimiento de un sistema invariable en el espacio. Madrid, La Universitaria, [1903]. 4to. 51 pp.

ROUBAUDI (C.). Cours de géométrie descriptive pour l'enseignement secondaire. 2e édition, entièrement refondue, conforme au programme du 31 mai 1902. Fascicule 1er: Second cycle. Classe de première (sections C et D). Paris, Masson, 1904. 8vo. 6 + 117 pp.

SCHILL (R.). Maturitätsaufgaben aus der darstellenden Geometrie nebst vollständigen Lösungen. Für die oberen Klassen der Realschulen und verwandter Anstalten sowie für das Selbststudium zusammengestellt und gelöst. Teil I: 140 Aufgaben. Wien, Deuticke, 1904. 8vo. 5 + 72 pp., 31 plates. M. 2.00

SCHLICK (O.). Die Untersuchung der Vibrationserscheinungen von Dampfern, an einem Beispiel erläutert. Leipzig, Felix, 1903. 8vo. 49 pp., 4 plates. M. 2.80

SCHLINK (W.). Stabilitäts- und Spannungsuntersuchungen von speziellen Fachwerksträgern mittels des erweiterten Systems. Habilitationsschrift zur Erlangung der venia legendi für Mechanik der allgemeinen Abteilung an der Grossherzoglich technischen Hochschule zu Darmstadt. Bonn, 1903. 8vo. 36 pp., 12 plates.

SCRIBANTI (A.). Lezioni sui calcoli relativi alla robustezza longitudinale degli scafi. Genova, 1903. 4to. Fr. 5.00

SKOUGE (D.). See OSTENFELD (A.).

SOMMERFELD (A.). See KLEIN (F.).

STÄCKEL (P.). Bericht über die Mechanik mehrfacher Mannigfaltigkeiten. (*Jahresbericht der Deutschen Mathematiker-Vereinigung*, Band 12 (1903), pp. 469–481.)

STODOLA (A.). Die Dampfturbinen und die Aussichten der Wärmekraftmaschinen. Versuche und Studien. Berlin, Springer, 1903. 8vo. 8 + 220 pp., 1 plate. Cloth. M. 6.00

TEISSERENC DE BORT (L.). See HILDEBRANDSSON (H.).

THURSTON (R. H.). A treatise on friction and lost work in machinery and millwork. 7th edition, enlarged. New York, Wiley, 1903. 8vo. 12 + 430 pp. Cloth. $3.00

VERSLUYS (J.). Beschrijvende meetkunde. 3ter druk. Deel II. Amsterdam, 1903. 8vo. 77 pp. M. 2.50

VILLETTE (T.). Eléments de géométrie pratique et de dessin linéaire. 6e édition. Paris, 1903. 8vo. 176 pp. Fr. 2.00

WAGSTAFF (C. J. S.) and BLOOMER (G. C.). Mensuration, mechanics, and hydrostatics. London, 1903. 4to. Boards. 1s. 6d.

WIEGHARDT (K.). Ueber die Statik ebener Fachwerke mit schlaffen Stäben. (Diss.) Göttingen, 1903. 8vo. 86 pp.